C1NOTES

A Calculus and MAPLE 7 Supplement

SM121 Fall, 2001-2002

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Calculus I | **C1Notes** | **Fall, 2001–2002**

by Professor C. E. Moore

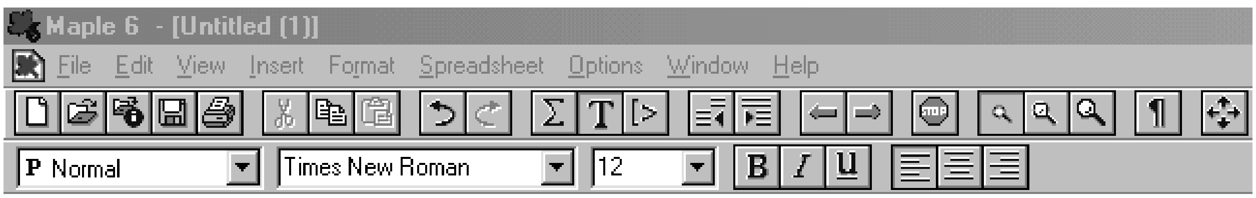
C1M0

Introduction to Maple

Our discussion will focus on Maple 7, which was developed by Waterloo Maple Inc. in Waterloo, Ontario, Canada. Quoting from the *Maple 7 Learning Guide*, "Maple is a *Symbolic Computation System* or *Computer Algebra System*. Both phrases refer to Maple's ability to manipulate information in a symbolic or algebraic manner. Conventional mathematical programs require numerical values for all variables. By contrast, Maple maintains and manipulates the underlying symbols and expressions, as well as evaluates numerical expressions." From the second description you see why Maple is designated as a 'CAS'.

Assignment Format

We are going to begin by establishing a format for each Maple assignment that is to be handed in. Open Maple and obtain a blank worksheet. Do not type the "<" or ">" which are shown to identify your entries. And <Enter> means the "Enter" key. As you begin, the worksheet is in "math mode", so 'click' on the $\boxed{\mathbf{T}}$ to switch into "text mode".



For the assignment **C1M1**, type <C1M1> <Enter> and then highlight C1M1 and click on the middle of the three boxes to the right of $\boxed{\mathbf{B}}$ $\boxed{I}$ $\boxed{\underline{u}}$ so as to center C1M1. The left of these three buttons left-justifies text and the right one right-justifies it. Now,
<down arrow>, then type your name and section as shown.
<Midn Your Name> <Enter>
<Section> <Your section> <Enter>
Having completed this, highlight the three lines and then click on $\boxed{\mathbf{B}}$ to boldface everything. This is the format you should use for all Maple assignments to be handed in. For example, you should see something like

C1M1

Midn John Doe
Section 1234

Beginning Maple Syntax

Since we are building the foundation for the use of Maple, we will designate this spadework by ♠ when we wish to highlight an important fact. Since getting on-line help is extremely important, we will begin with that. Suppose that you have a question about the command 'plot'. Then in a worksheet enter <?plot> <Enter> and you will see the information available and links for other related topics.

♠ To obtain on-line help on 'command', enter <?command> <Enter>.

You may eliminate the brackets on the left by pressing the function key <F9>. To return to math mode, click on the $\boxed{[>}$. If we wanted to type a math formula while in text mode we would click on $\boxed{\Sigma}$. Later in this section we will discuss *palettes* which allow you to select commands from a menu and avoid using Maple syntax. It is the contention of the author of these notes that learning some Maple syntax is beneficial to the student, so even though you may accomplish the same things by clicking on a symbol, we will show you the syntax that would otherwise be hidden.

In math mode, lines in Maple end with a semicolon or colon.

♠ If a line ends with a semicolon, then the output will be displayed.

♠ If a line ends with a colon, then the display of the output is suppressed.

♠ To activate a line, press <Enter> with the cursor at **any** position on that line. (This does not break the line as it would in a word processor.)

Please type the command lines below in a new worksheet exactly as you see them, remembering to press <Enter>, and note the output. This work is for your benefit and is not intended to be handed in.

```
> a:=4;
> sqrt(a);
> b=4;
> sqrt(b);
```

We did not display the output here because it is important that the reader discover the results for themself. However, the concept is very important. The first line shows the format for assigning a value to the variable 'a'. Think of this as placing the value 4 in a memory location, named 'a', which can be retrieved when needed. The third line is an equation. While it is trivial to solve for b, the value of b is not accessible in this format.

♠ Use `a:=b` to assign the value 'b' to 'a'. Note that 'b' may be a number or an expression.

Expression versus Function

Please enter the following lines in a worksheet, remembering to press <Enter> to activate each line:

```
> A:=x^2+sin(4*x);
> f:=x->sqrt(x^2+9);
> subs(x=4,A);
> f(x); f(4);
> subs(x=4,f(x));
> simplify(%);
```

The first line names the **expression** $x^2 + \sin(4x)$ as 'A'. Note how '$*$' must be used when two factors are to be multiplied. Omitting the $*$ is a common error for beginning Maple users. The second line shows how to define a **function** f. The third line shows how to substitute 4 for x in the expression A. Note: You may **not** use $4 = x$ and expect the same result. This part of the substitution must be of the form 'old' = 'new', because the 'old' will be replaced by the 'new'. Try reversing the order of the substitution by entering <`subs(4=x,A);`>. What difference do you see?

The fourth line above is more important than it appears to be. You see that $f(x)$ is an **expression** and then the fifth line reinforces that. However, were you surprised at the output of the fifth line? There are many levels of operations and simplifications in Maple and substitution does not simplify as far as is possible. The use of % as the `simplify` argument refers to the previous output, which was $\sqrt{25}$.

♠ You must be aware of when the syntax calls for an expression or for a function.

♠ When two factors are to be multiplied, a '$*$' must separate them, i.e. `a*b`.

♠ You may use `%` to refer to the previous output in a Maple command. Warning! See comments about order of execution later in this section.

In that same worksheet enter these command lines and observe the output.

```
> value(Pi^2/6 + sin(Pi/6));
```

$$\frac{1}{6}\pi^2 + \frac{1}{2}$$

```
> evalf(Pi^2/6 + sin(Pi/6));
```

$$2.144934068$$

```
> evalf(Pi^2/6 + sin(Pi/6),75);
```

$$2.14493406684822643647241516664602518921894990120679843773555822937000747041$$

The '`value`' command returns an exact value for an expression, while the `evalf` command converts an exact numerical expression to a floating point number. You also have the option of specifying the number of digits displayed, as shown on the third line. The default is to display ten digits. You may also set the number of digits to be displayed in the remainder of a worksheet by using a command, for example:

```
> Digits:=20;
```

$$Digits := 20$$

```
> evalf(Pi);
```

$$3.1415926535897932385$$

```
> evalf(pi);
```

$$\pi$$

♠ In Maple, `Pi` $= \pi$ is a number, and `pi` $= \pi$ is a small Greek letter with no numerical value.

It will be very useful later to be able to make a function out of an expression. The syntax for this is puzzling. Entering these commands should produce these results:

```
> P:=x^2+cos(x);
```

$$P := x^2 + \cos(x)$$

```
> G:=unapply(P,x);
```

$$G := x \to x^2 + \cos(x)$$

```
> G(Pi);
```

$$\pi^2 - 1$$

We see that G is a function and that $G(x) = P$.

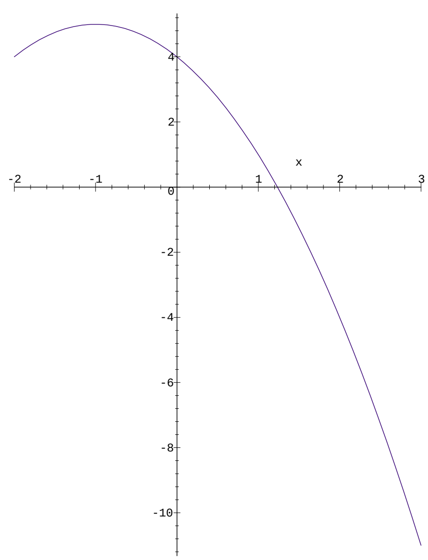
Maple Graphics

Maple graphics are versatile and easy to use. Let's define $F(x) = 4 - 2x - x^2$ in our worksheet and see how we can get a quick plot of F on $[-2, 3]$. To save space we have included two plots side-by-side. The output of the first is on the left.

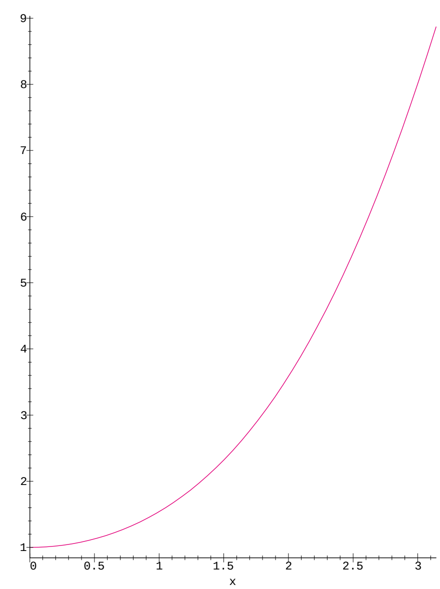
```
> F:=x->4-2*x-x^2;
```

$$F := x \to 4 - 2x - x^2$$

```
> plot(F(x),x=-2..3,color=blue);
```



Plot of F(x)



Plot of P

Note how we used $F(x)$ which is an expression, not just F, in the first plot. Now click on the displayed graph. A box will appear around the graph with small black boxes placed strategically. Move the cursor over the lower righthand box and position it so that a diagonal arrow appears. Click when the arrow is displayed and drag the arrow towards the center of the box, thereby resizing the box. You can change the aspect of the graph by making the box tall or short. You should always make the plots on your homework smaller so as to save paper.

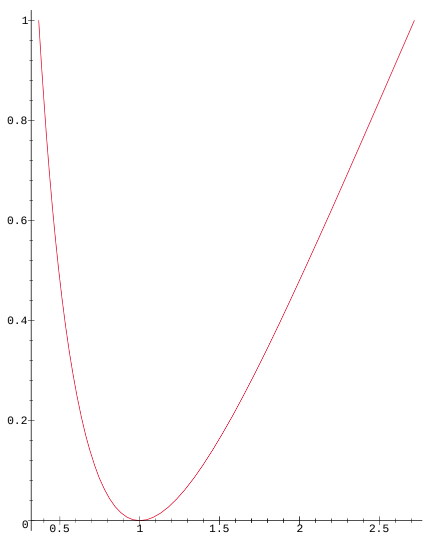
♠ You can resize a Maple plot by clicking on the plot and then dragging the corner of the displayed box.

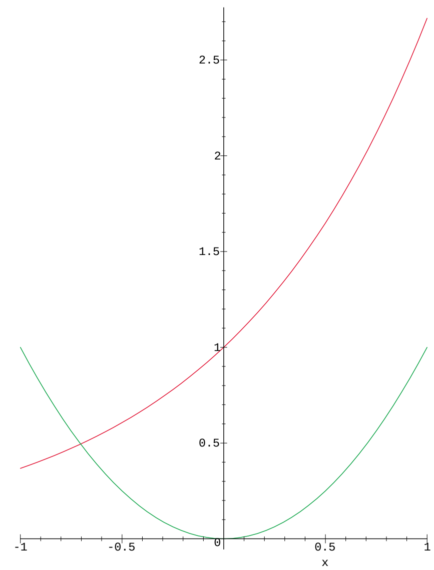
To plot the function G from above we could use P or $G(x)$ and obtain identical results. The output is on the right above.

```
> plot(P,x=0..Pi,color=magenta);
```

When you wish to plot two functions with the same domain it can be done very easily. However, it is also very easy to confuse this syntax with that of parametric plotting. We will do an example of each so that you will know where to be careful. The placement of the righthand square bracket determines which format you have. In two-dimensional plotting, when you list two expressions and a range inside the square brackets the first function controls the value on the x-axis and the second function controls the value on the y-axis. This is parametric plotting. To save space the output follows on the left.

```
> plot([exp(x),x^2,x=-1..1]);
```





Parametric Plotting **Two Functions**

When you do not include the domain inside the square brackets you get two different plots on the same coordinate system as you can see above on the right. This was produced by:

```
> plot([exp(x),x^2],x=-1..1);
```

♠ To plot the graphs of two functions with the same domain on the same axes, include both functions in square brackets, but exclude the range. If the range is included in the brackets, then the result will be a parametric plot.

♠ The exponential function, e^x, is accessed by `exp(x)` in Maple, which does not recognize e as any particular number. See the following:

```
> evalf(e);
```

$$e$$

```
> evalf(exp(1),90);
```

2.71828182845904523536028747135266249775724709369995957496696762772407663035354759457138217

Maple Packages

Just as specialized mechanics require special tools, and they do not carry every trade's tools with them at all times, Maple has packaged different commands into different libraries so that not all commands need to be put into active memory at all times. Instead, the user may select one or more packages as needed and thereby save computer memory space. For a list of all the packages, see section **3.8** on page 104 of the *Maple 7 Learning Guide*. Or, in a worksheet type <?packages> to see the listing in the *Help* section of Maple.

There are several packages that the beginning calculus student will need. The first is `student`, which contains many calculus operations, and the second is `plots`. As you would expect, `plots` is a graphics package. To invoke, or 'call up' a package you would insert a line in the worksheet before the package is needed such as:

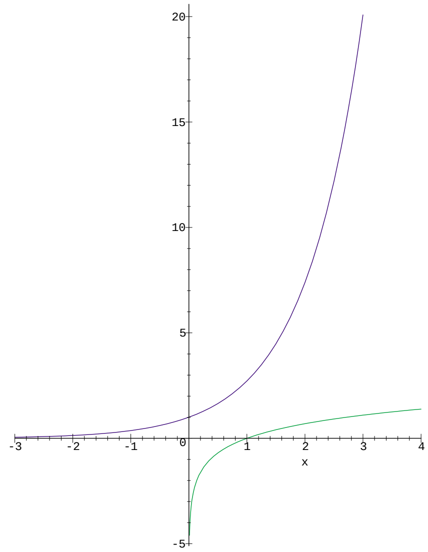
```
> with(plots):
```

If you had ended the line above with a semi-colon, then a list of all the commands in `plots` would have been displayed. You should probably try this once just to see what is there. Also, try putting a semi-colon after `with(student)`.

More Graphics

How should you plot two functions which have different domains on the same coordinate axes? The answer is to give each plot a name **using a colon at the end to suppress the output** and then display them together. The command `display` is in the graphics package `plots`.

```
> with(plots):
> A:=plot(exp(x),x=-3..3):    ← colon!
> B:=plot(ln(x),x=.01..4):    ← colon!
> display(A,B);    ← semicolon!
```



Order of Execution of Commands

Sometimes previous experience with computers causes us to assume that the order of execution of the commands in Maple is the same as the order shown on the screen. Also, we assume that whatever we see in front of us has been executed. Both assumptions are false in Maple. Suppose that you have a worksheet that you had saved previously and that you have just reopened it. You move the cursor down to some line in the worksheet and hit <Enter>. As far as Maple is concerned, this is the first line executed and if it depends on lines above it, then it cannot execute correctly. Computationally, Maple remembers the information in the order in which the lines were executed.

Also, sometimes we want Maple to forget that it has done something and to start over. For this reason the first command line should have `restart:` as the first word. Then, call up the packages that you anticipate using. For example:

```
> restart:   with(student):   with(plots):
```

is a typical first command line in Calculus I. Since `restart:` clears the memory, what would happen if it occurred last on the line? That's right, the packages would be loaded and then erased from accessible memory.

It is quite reasonable and normal to move around in a worksheet, changing things as needed. In fact, one frequently realizes that a new command line needs to be inserted earlier in a worksheet. To do this, place the cursor on the line above where you want a new line. Then, click on the $\boxed{[>}$ button and a new blank command line will appear. But, when you execute this line by hitting <Enter>, what will be the order of execution of the Maple commands? How do you ensure that the order of execution of the commands is the same as what you see? One approach would be to put the cursor on the first line and hit <Enter> until you reach the last line. That is effective, but it is not necessary. Move the cursor up to 'Edit' on the left at the top of the screen, 'click', and then slide the cursor down to 'Execute'. A side panel will open, slide the cursor over and down to 'Worksheet', and click'. At this point, Maple will execute the worksheet in the order of the commands shown on the screen.

♠ To insert a new command line in a worksheet, put the cursor on the line above and click on $\boxed{[>}$.

♠ **Always** execute a worksheet upon reopening it and before additional commands are added.

♠ **Always** execute a worksheet before saving it and printing it out to be handed in.

Example: Suppose that we are given two points in the plane, $P_1(2,5)$ and $P_2(-1,1)$, and we wish to find the distance between them and an equation for the line that contains them.

```
> restart:
> x1:=2; y1:=5; x2:=-1; y2:=1;
```

$$x1 := 2$$
$$y1 := 5$$
$$x2 := -1$$
$$y2 := 1$$

```
> distance:=sqrt((x2-x1)^2+(y2-y1)^2);
```

$$distance := 5$$

```
> slope:=(y2-y1)/(x2-x1);
```

$$slope := \frac{4}{3}$$

```
> line1:=y-y1=slope*(x-x1);
```

$$line1 := y - 5 = \frac{4}{3}x - \frac{8}{3}$$

```
> y=solve(line1,y);
```

$$y = \frac{7}{3} + \frac{4}{3}x$$

The process is straightforward until we reach the line that begins with `line1`. We assigned the name ‘line1’ to the equation for the line using the format $y - y_0 = m(x - x_0)$, where m is the slope of the line and $P(x_0, y_0)$ is a known point on the line. We chose to use P_1, but the choice of P_2 would have worked equally well. Then, we established a new equation with y being set equal to the solution of the equation $line1$ for the variable y. This yields the format $y = mx + b$, which some prefer.

There are two operations that are very basic in calculus, namely differentiation and integration, or anti-differentiation. An expression in x and t, say $\tan(x/t)$, can be differentiated with respect to either variable, so we must remember to specify the variable with respect to which the operation is being performed. Using the expression $P = x^2 + \cos(x)$ from above we have

```
> Pprime:=diff(P,x);
```

$$Pprime := 2x - \sin(x)$$

And if we integrate P

```
> Pint:=int(P,x);
```

$$Pint := \frac{x^3}{3} + \sin(x)$$

Now let’s do some of the same steps by using a palette. On your command line type <A:=> to get

```
> A:=
```

There are three palettes and to access them you begin by clicking on “View”, then “Palettes”. If you need symbols, select that palette, but for now we choose “expressions”. You should see

Click on the box with the integral symbol $\boxed{\int a}$. Then click on the box with $\boxed{a^b}$. On your command line the cursor appears where you want x inserted, so you type $< x >$, and then **move to the next entry position by using the ‘Tab’ key**. Enter $< 2 >$, <Tab>, and then the variable of integration, $< x >$ and <Enter>. At the end of the command line put a semicolon and hit <Enter>. This should produce

```
> A:=int(x^2,x);
```

$$\frac{x^3}{3}$$

It is not the purpose of this section to teach you about differentiation or integration. That will come later. But, you see how that palettes can be used to accomplish these tasks.

C1M0 Exercises: Use Maple to find the plots and answers.

1. Display the graph of $y = 3\sin(2x + \pi/6) + 2$ on the interval $\left[-\pi, \frac{3\pi}{2}\right]$.

2. Define $f(x) = x^2 - 4x + 4$ and $g(x) = \ln(x + 2)$ as functions and in the same plot display their graphs for the interval $[-1, 4]$.

3. Find the value of e^{-2x} for $x = .7$ and display 25 places.

C1M1

New Functions from Old Functions

We will approach this topic by providing examples and plots of functions. First, we will demonstrate some definitions and related Maple syntax by establishing some equations in a worksheet. The reader might benefit by typing in each line or copying and pasting the commands from **C1Notes** as found on the syllabus webpage that the instructor provided.

```
> restart:
> f:=x->x^2-x+1;   g:=x->sin(2*x);
```

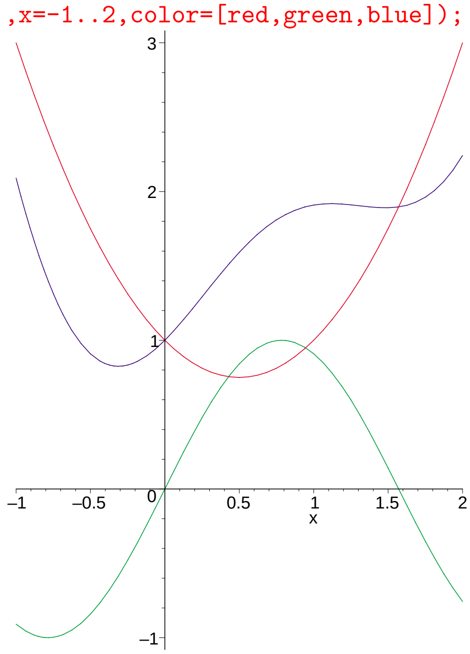
$$f := x \to x^2 - x + 1$$
$$g := x \to \sin(2x)$$

Addition

```
> (f+g)(x)=f(x)+g(x);
```

$$x^2 - x + 1 + \sin(2x) = x^2 - x + 1 + \sin(2x)$$

```
> plot([f(x),g(x),(f+g)(x)],x=-1..2,color=[red,green,blue]);
```



Subtraction

```
> (f-g)(x)=f(x)-g(x);
```

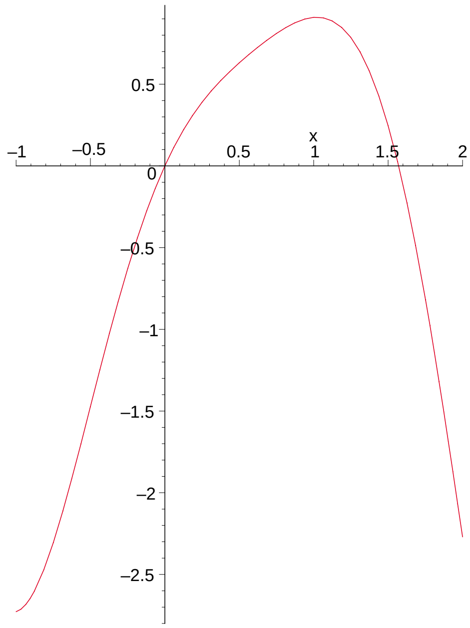
$$x^2 - x + 1 - \sin(2x) = x^2 - x + 1 - \sin(2x)$$

Multiplication

```
> (f*g)(x)=f(x)*g(x);
```

$$(x^2 - x + 1)\sin(2x) = (x^2 - x + 1)\sin(2x)$$

```
> plot((f*g)(x),x=-1..2);
```

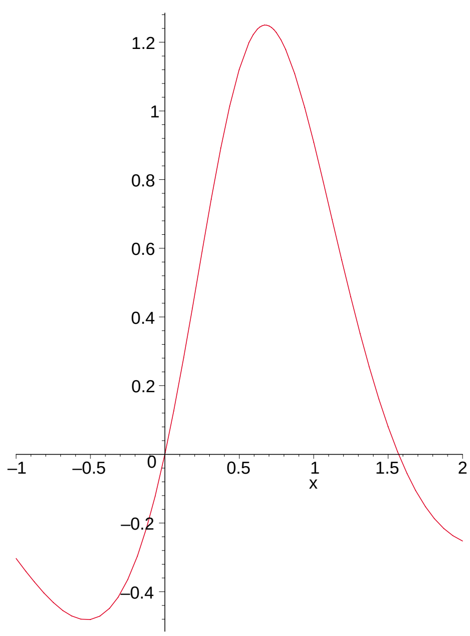


Division

```
> (g/f)(x)=g(x)/f(x);
```

$$\frac{\sin(2x)}{x^2-x+1} = \frac{\sin(2x)}{x^2-x+1}$$

```
> plot((g/f)(x),x=-1..2);
```

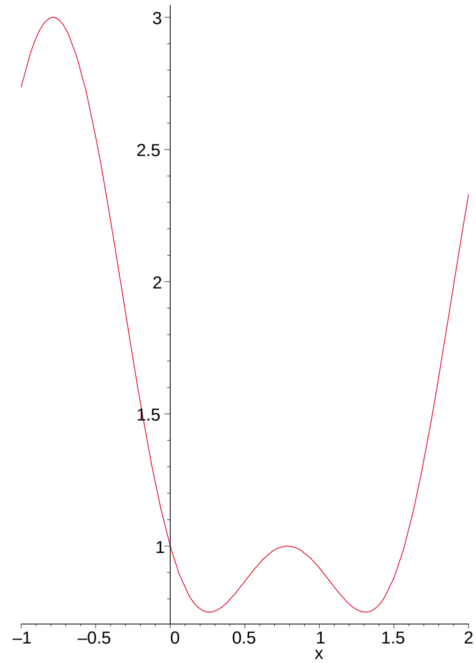


Composition

```
> (f@g)(x)=f(g(x));
```

$$\sin(2x)^2 - \sin(2x) + 1 = \sin(2x)^2 - \sin(2x) + 1$$

```
> plot((f@g)(x),x=-1..2);
```

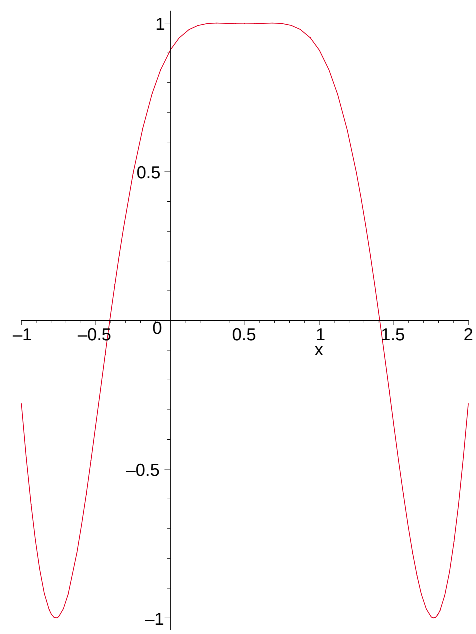


Composition

```
> (g@f)(x)=g(f(x));
```

$$\sin(2x^2 - 2x + 2) = \sin(2x^2 - 2x + 2)$$

```
> plot((g@f)(x),x=-1..2);
```



Maple Example For the function $f(x) = x + \sin(x)$, find all values of x for which $f(x + \pi/6) = f(x - \pi/6) + 1$.

```
> f:=x->x+sin(x);
```

$$f := x \longrightarrow x + \sin(x)$$

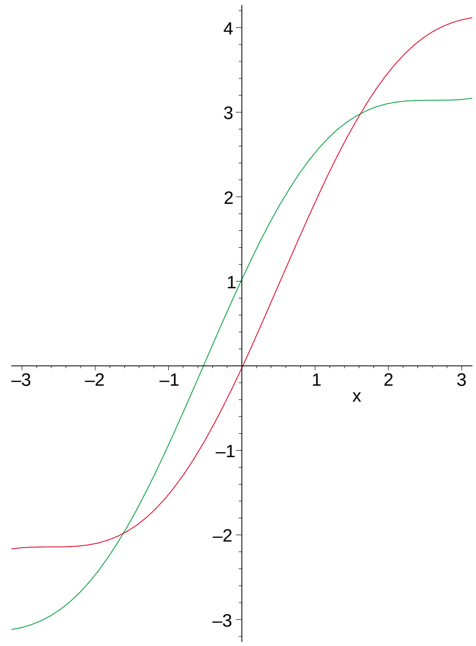
```
> eq1:=f(x-Pi/6)+1=f(x+Pi/6);
```

$$eq1 := x - \frac{\pi}{6} - \cos\left(x + \frac{\pi}{3}\right) + 1 = x + \frac{\pi}{6} + \sin\left(x + \frac{\pi}{6}\right)$$

```
> evalf(solve(eq1,x));
```

$$-1.618011419,\ 1.618011418$$

```
> plot([f(x-Pi/6)+1,f(x+Pi/6)],x=-Pi..Pi);
```



We see from the graph that the curves cross at two points, so we found all of the solutions using Maple. This is not always the case, especially when there are an infinite number of solutions. There is another Maple command, `fsolve`, that returns a floating point number and will permit one to limit the range where a solution is being sought. This next example illustrates this.

Maple Example: Suppose that $h(x) = -\frac{x}{2} + 2$ and $j(x) = \sin(x)$, find all solutions of $h(x) = 2\,j(x)$.

```
> h:=x->-x/2+2;  j:=x->sin(x);
```

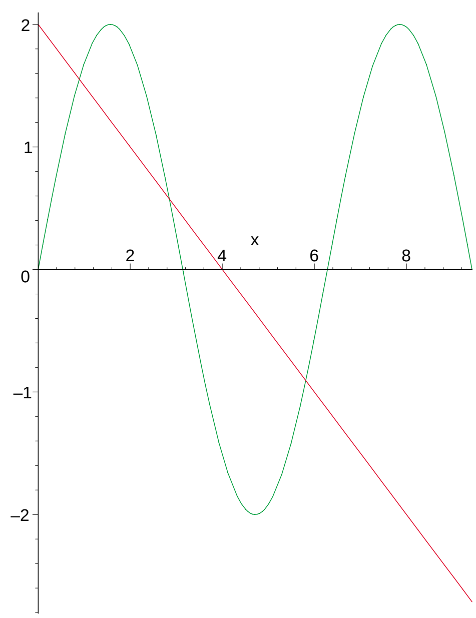
$$h := \to -\frac{x}{2} + 2$$
$$j := \sin$$

```
> solve(h(x)=2*j(x),x);
```

$$\mathrm{RootOf}(\_Z - 4 + 4\sin(\_Z))$$

```
> evalf(%);
```

$$.8904870807$$



By using `evalf` we get a numerical answer for our solution. But, we suspect that other solutions might exist so we plot the graphs of each side of our equation. We see that there are two more solutions, one between 2 and 4, the other between 4 and 6.2. The number 6.2 is slightly larger than 2π, where $\sin(x)$ has a zero and the intersection is obviously before that. We turn to fsolve and select a range for the value, which fsolve permits. However, fsolve seeks a numerical solution and returns a single floating point number, not more.

```
> plot([h(x),2*j(x)],x=0..3*Pi);
```

```
> fsolve({h(x)=2*j(x)},{x},2..4);
```

$$\{x = 2.849968934\}$$

```
> fsolve({h(x)=2*j(x)},{x},4..(6.2));
```

$$\{x = 5.812826090\}$$

Note that when we use set notation in evalf, Maple returns the output in similar notation.

C1M1 Problems: Use Maple to plot the graphs to see where they intersect, and to find the solutions.

1. For $f(x) = \frac{x}{2} + 2$ and $g(x) = 2^x$, find all solutions of $f(x) = g(f(x-2))$.

2. For $p(x) = \sin(x/6)$ and $q(x) = \sqrt{x^2+4}$, find all solutions of $p(q(x)) - x^2 = x$ for $-2 \le x \le 2$.

3. For $r(x) = \cos(x)$, find all solutions of $(r \circ r)(x) = x$.

C1M2

Parametric Curves

Have you ever played with a toy called *"Etch-a-Sketch"*? One hand controls the x-axis while the other controls the y-axis. It is as if you are graphing $(x(t), y(t))$, $a \le t \le b$, which is exactly what happens when a function in the plane is defined parametrically. Be very careful where you place the right bracket,], when using Maple to plot parametric curves.

Relationships between trigonometric functions, and in particular the Pythagorean identities, are used quite often when eliminating the parameter and thereby identifying the curve that has been described parametrically. For your convenience, we provide the Pythagorean identities:

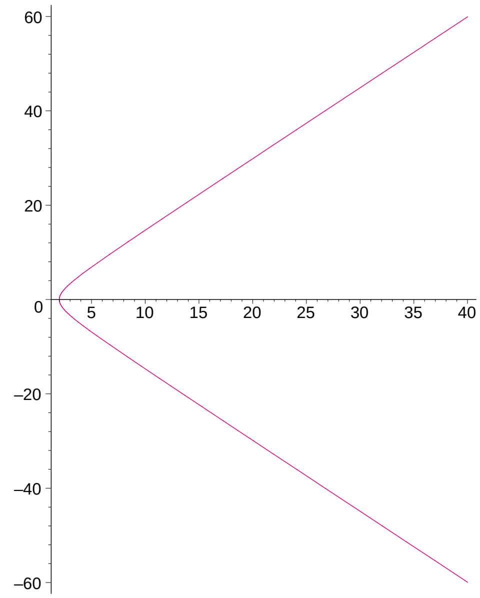
$$\sin^2(\theta) + \cos^2(\theta) = 1 \qquad \tan^2(\theta) + 1 = \sec^2(\theta) \qquad 1 + \cot^2(\theta) = \csc^2(\theta)$$

Maple Example: Plot $x(t) = 2\csc(t)$, $y(t) = 3\cot(t)$ for $0 < t < \pi$ and eliminate the parameter. We solve for the trigonometric functions first and then apply the third identity.

$$\csc(t) = \frac{x}{2}, \quad \cot(t) = \frac{y}{3} \quad \Rightarrow \quad 1 + \frac{y^2}{9} = \frac{x^2}{4} \quad \Rightarrow \quad \frac{x^2}{4} - \frac{y^2}{9} = 1$$

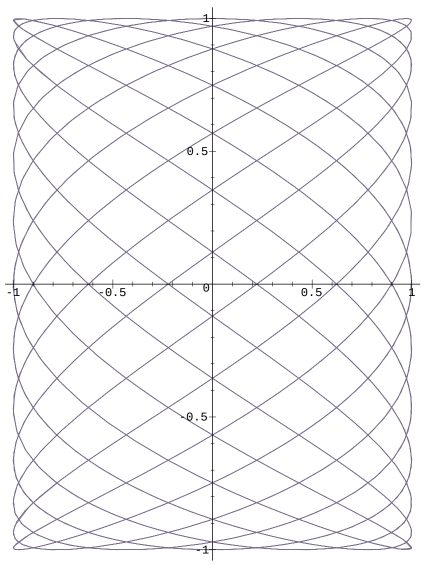
We see that our curve is a hyperbola and we check out the domain and find that not only must x always be positive, it must always be greater than 2. Because we must avoid the endpoints of the domain, note how this is done in our plot. Is it easy to see from the plot that the hyperbola is asymptotic to the straight lines $y = \pm\frac{3}{2}x$?

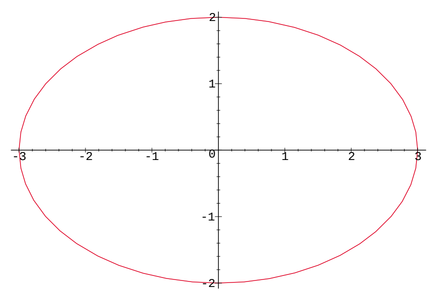
```
> plot([2*csc(t),3*cot(t),t=.05..Pi-.05],color=magenta);
```



Maple Example: Plot $x(t) = \sin(13t)$, $y(t) = \cos(7t)$ for $0 \le t \le 6\pi$ which produces a *lissajou*. The plot follows on the left. As you can see, the scaling is a little off because the "square" is two units on each side. For a little fun, increase the coefficients to say 43 and 37 and see what happens.

```
> plot([sin(13*t),cos(7*t),t=0..6*Pi],color=navy);
```





Maple Example: Ellipses are easy this way. Plot $\dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} = 1$. The Maple output is above on the right. When you have $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ you may plot this by using $x(t) = a\,\cos(t)$ and $y(t) = b\,\sin(t)$ for $0 \le t \le 2\pi$.

```
> plot([3*cos(t),2*sin(t),t=0..2*Pi]);
```

C1M2 Problems: Use Maple to display the parametric graphs of the given functions. Reminder:

♠ The exponential function, e^x, is accessed by `exp(x)` in Maple.

1. $x = e^t,\ y = e^{2t},\ -1 \le t \le 2$
2. $x = 2\sec t,\ y = \tan t,\ -\pi/2 < t < \pi/2$
3. $x = t - \sin t,\ y = 1 - \cos t,\ \ 0 \le t \le 4\pi$
4. $x = \cos^3 t,\ y = \sin^3 t,\ 0 \le t \le 2\pi$

C1M3

Exponential Functions

Every thirty days a biology student counted the number of bugs that lived in a colony and recorded the numbers, which we list for you.

124, 248, 496, 992, 1984

The student observed that the population was actually doubling each time, so the same data could be recorded as:

$124,\ 124 \cdot 2,\ 124 \cdot 2^2,\ 124 \cdot 2^3,\ 124 \cdot 2^4$

And since the data was collected every 30 days, the following formula was proposed as a model that represented the population each day, where t is the number of days.

$$P = 124 \cdot 2^{t/30}$$

Using this, the number of bugs after 47 days was estimated to be $124 \cdot 2^{47/30} \approx 367.3$. And, after 12 thirty day periods there would be $124 \cdot 2^{360/30} = 507,904$ bugs.

The idea of doubling something and recording the result is not a new one. Imagine putting a penny on the corner square of a chessboard and having someone put double the previous amount on a square for the next 63 days. How much would be accumulated? The amount placed each day, where you put the penny down on the 't^{th}' day would be

$$Q(t) = (.01)2^t$$

This means that $\$1,342,177.28$ – over a million dollars, would be placed on the 27^{th} day and the total after all 64 squares are accounted for would be approximately $1.844674407 \times 10^{17}$ dollars. This is roughly equivalent to 184,467 trillion dollars.

Suppose that we consider a function $f(x) = a^x$ for some $a > 0$. It is interesting to note that for any two values of x that are a fixed distance apart, like x and $x + h$, you get the same ratio

$$\frac{f(x+h)}{f(x)} = \frac{a^{x+h}}{a^x} = a^h = \text{constant}$$

for their functional values. When the first example involving the bugs was constructed it was decided that it was easiest to double the number of bugs every 30 days. But, suppose that the bugs increased by a factor of 1.375 every 30 days. Then we would have gotten $P = 124 \cdot (1.375)^{t/30}$ as our population function.

Maple Example: Suppose that a disease hits a flock of critters. Every week a count is taken to see how many remain. It is suspected that the relationship is exponential. If it is, find an expression which provides the number present at any time t in weeks. The table shows the data.

Week	Critters
0	1165
1	728
2	455
3	284
4	177
5	111
6	69

We begin by establishing a list of our data. Then, we check out the ratios of successive terms by using the `seq` command, which yields a list for each integer that is specified. The `[i]` refers to the i^{th} member of the list *dataset*. For example, `dataset[3]=455`.

```
> dataset:=[1165,728,455,284,177,111,69];
```

$$dataset := [1165, 728, 455, 284, 177, 111, 69]$$

```
> seq(dataset[i]/dataset[i-1],i=2..7);
```

$$\frac{728}{1165}, \frac{5}{8}, \frac{284}{455}, \frac{177}{284}, \frac{37}{59}, \frac{23}{37}$$

In Maple, you may use `%` to refer to the result(s) of the last command executed. It is extremely useful, but **BEWARE!** if you skip around in a worksheet because the order of execution of commands can produce some erroneous results. That is why it is important to 'execute the worksheet' before you hand it in. Then, it is important to look back over it to see if errors have occurred. After examining the results, we select $a = .625$ as a potential base.

```
>  evalf(%);
```

$$.6248927039, .6250000000, .6241758242, .6232394366, .6271186441, .6216216216$$

```
>  A0:=1165;  a:=.625;
```

$$A0 := 1165$$
$$a := .625$$

```
> Q:=t->A0*(a)^t;
```

$$Q := \rightarrow A0\, a^t$$

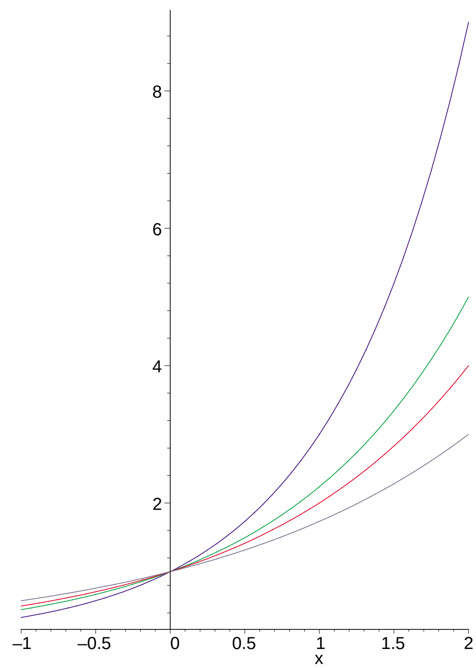
```
> seq(Q(i),i=0..6);
```

$$1165., 728.125, 455.078125, 284.4238281, 177.7648925, 111.1030579, 69.43941117$$

The function Q that is defined seems to reflect the number of critters very well.

It might be interesting to look at functions of the form $y = a^x$ for different positive values of a.

```
> plot([sqrt(3)^x,2^x,sqrt(5)^x,3^x],x=-1..2,color=[navy,red,green,blue]);
```

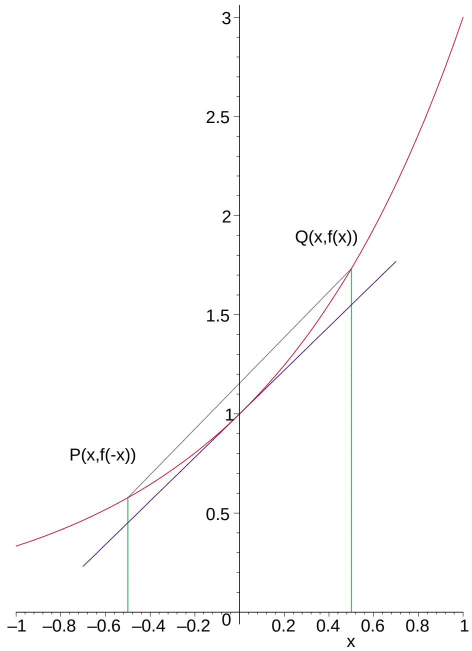


We recall that $\sqrt{3} < 2 < \sqrt{5} < 3$, so that when $0 < x$ the graphs maintain the related inequalities, $(\sqrt{3})^x < 2^x < (\sqrt{5})^x < 3^x$. But wait!, It gets better. In a worksheet, enter the following three lines and execute them:

```
> with(plots):
> G:=a^x;
> animate(G,x=-1..2,a=1/2..4);
```

And what will appear? You will obtain the expression G and a plot which shows $y = \left(\frac{1}{2}\right)^x$. Move the cursor to the plot and 'click' on it. The *toolbar* at the top of the screen, which is below the *menu bar*, will be replaced by what looks like the buttons on a tape player. Click on the triangular 'play' button. What happens? You should see G plotted for values of the parameter a as a varies from $1/2$ to 4.

From this animation we see that there should be some value of a for which the slope of a^x at $x = 0$ should equal 1. We will use brute force and admittedly prior knowledge to try to find that special value of a to get a slope closer and closer to 1. We begin by defining a function F that produces the slope of the line that joins the two points $P(-x, a^{-x})$ and $Q(x, a^x)$ spaced equally from $x = 0$. As x gets smaller, these two points get closer together and the slope determined should be closer to that of a line tangent to the graph of $y = a^x$ for $x = 0$.



The `seq` command produces a sequence of values of F with $x = \dfrac{1}{2^i}$ for $i = 3$ to $i = 10$.

It is easier to obtain the lines below than it looks. Once you have entered the command line where `a:=0`, simply copy and paste to lines below, change the value of a, and hit <Enter>.

```
> F:=x->(a^x-a^(-x))/(2*x);
```

$$F := x \longrightarrow \frac{1}{2}\frac{a^x - a^{-x}}{x}$$

```
> a:=2;  seq(evalf(F(1/2^i)),i=3..10);
```

$$a := 2$$

$$.694014760, .693364012, .69320139, .69316073, .69315056, .6931480, .6931474, .6931474$$

```
> a:=3;  seq(evalf(F(1/2^i)),i=3..10);
```

$$a := 3$$

$$1.102068589, 1.099475751, 1.09882812, 1.09866622, 1.09862576, 1.0986156, 1.0986131, 1.0986123$$

```
> a:=2.7;  seq(evalf(F(1/2^i)),i=3..10);
```

$$a := 2.7$$

$$.9958055410, .9938898465, .9934112640, .9932916545, .9932617665, .9932542335, .9932524545, .9932519425$$

```
> a:=2.7183;  seq(evalf(F(1/2^i)),i=3..10);
```

$$a := 2.7183$$

$$1.002612939, 1.000657870, 1.000169456, 1.000047386, 1.000016870, 1.000009280, 1.000007398, 1.000006810$$

```
> a:=2.718281828;  seq(evalf(F(1/2^i)),i=3..10);
```

$$a := 2.718281828$$

$$1.002606202, 1.000651170, 1.000162760, 1.000040704, 1.000010157, 1.000002496, 1.000000742, .9999998975$$

```
> a:=exp(1);  seq(evalf(F(1/2^i)),i=3..10);
```

$$a := \mathrm{e}$$

$$1.002606202, 1.000651170, 1.00016276, 1.00004071, 1.00001016, 1.0000025, 1.0000007, .9999999$$

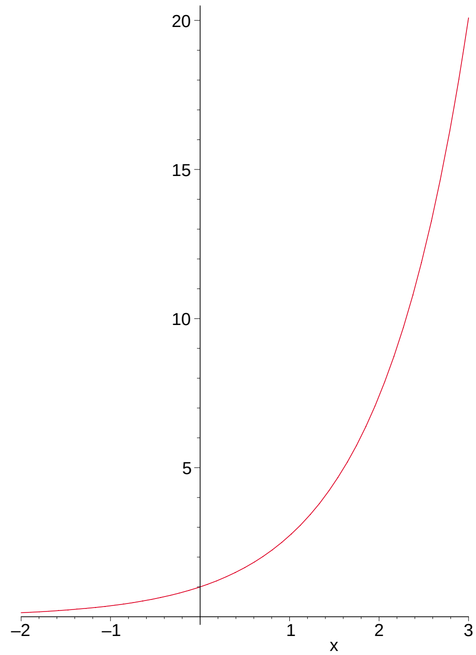
In the **C1M0** – *Introduction to Maple*, we showed e or `exp(1)` to 90 places. There are several ways to define e, but for now it suffices to regard it as that number a for which $y = a^x$ has a slope of 1 at $x = 0$. The function $y = e^x$ occurs frequently in growth and decay problems and its importance to Calculus and the Sciences cannot be overstated. Frequently students underestimate how rapidly e^x grows as x increases. For this reason and the fact that $e^3 = 20.08553692\ldots$ we provide this 'Rule of Thumb':

$$e^3 \approx 20$$

Think about it! This means that $e^6 = (e^3)^2$ must be over 400 and e^9 must be over 8000. After we provide the following reminder, we will plot e^x.

♠ The exponential function, e^x, is accessed by `exp(x)` in Maple, which does not recognize e as any particular number.

```
> plot(exp(x),x=-2..3);
```



Maple Example: It is known that radioactive materials satisfy an equation of the form $A = A_0e^{kt}$ where A_0 is the initial amount and k is a negative constant. Suppose we have 2.837 grams of an unstable compound whose half-life is 137 days. How much will be present after 97 days and when will we have exactly 1 gram remaining?

First, we must find the value of k, using the half-life of the compound.

```
> A0:=2.837;
```

$$A0 := 2.837$$

```
> A:=A0*exp(k*t);
```

$$A := 2.837e^{(kt)}$$

When $t = 137$, the amount present is one-half the original amount. We set this up as an equation, which we can solve for k.

```
> eq:=A0/2=subs(t=137,A);
```

$$eq := 1.418500000 = 2.837e^{(137k)}$$

```
> k:=solve(eq,k);
```

$$k := -.005059468471$$

```
> A;
```

$$2.837e^{(-.005059468471t)}$$

We see that Maple is using the value of k. Substituting $t = 97$ we find the amount at that time.

```
> A97:=subs(t=97,A);
```

$$A97 := 2.837e^{(-.4907684417)}$$

```
> A97:=evalf(A97);
```

$$A97 := 1.736686025$$

That gives the amount at 97 days. Now to find when there is one gram we must set the amount equal to 1, and solve for t.

```
> eq1:=1=A;
```

$$eq1 := 1 = 2.837e^{(-.005059468471t)}$$

```
> solve(eq1,t);
```

$$206.0981626$$

C1M3 Problems: Use Maple to solve the following problems.

1. Plot the graph of the expression A in the last example for $0 \le t \le 250$.
2. Plot e^x and $1 + x + x^2/2 + x^3/6$ on the same coordinate axes for $-3 \le x \le 3$.

3. The command `evalf(solve(2+x=exp(x)));` finds all solutions of the equation $2+x=e^x$. Use a similar command to find all solutions of $2x^2=e^x$.

4. Plot $2x^2$ and e^x on the same coordinate axes for $-1 \le x \le 3$.

5. The data shown may be related by an exponential function. Determine a function that fits this data.

Time	Reading
1.2	6.292161022
1.35	6.783009413
1.5	7.312148644
1.65	7.882565768
1.8	8.497480852
1.95	9.160365159

C1M4

Inverse Functions and Logarithms

Each summer a new group of incoming students is inducted into the U.S. Naval Academy, they become Fouth Class Midshipmen or *plebes*, and identification numbers called *alpha numbers* are assigned. Since this year is 2000, and it is hoped that these students will graduate in 2004, each number assigned begins with 04. So a typical alpha might be 047854, except that 7854 is a larger number than would be needed. Unless an error has been made, there is a $one-to-one$ relationship between a set called Plebes and a set called Alphas. If numbers are assigned to names then there is a function F so that

$$F : \mathit{Plebes} \longrightarrow \mathit{Alphas}$$

and, for example

$$F(\text{John Doe}) = 041721$$

And, unless two plebes are mistakenly assigned the same number, there is a unique association that allows one to identify the plebe if you have their alpha number. This means that there is an *inverse function*

$$F^{-1} : \mathit{Alphas} \longrightarrow \mathit{Plebes}$$

and

$$F^{-1}(041721) = \text{John Doe}$$

The discussion above is a very simplistic example of how functions and inverse functions relate. More precisely, with $\iff$ meaning "if and only if",

$$f^{-1}(x) = y \qquad \iff \qquad f(y) = x$$

You have studied exponential functions recently and probably noted that when $a > 1$, then a^x is an increasing function. We sometimes use $\equiv$ to denote a definition or an equivalent statement.

$$f \text{ is increasing} \quad \equiv \quad u < v \Rightarrow f(u) < f(v)$$

It is easy to see that when a function is increasing then it has an inverse. And, inverses of exponential functions are called *logarithms*. This leads to the very important relationship

$$\log_a(x) = y \qquad \iff \qquad a^y = x$$

In calculus, the most important base is e and we call that logarithm the *natural logarithm* and identify it by ln.

$$\log_e(x) \equiv \ln(x)$$

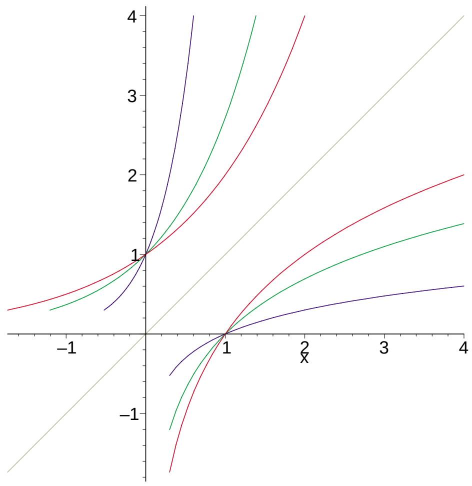
From which it follows that

$$\spadesuit \quad e^{\ln(x)} = x \qquad e^{-\ln(x)} = \frac{1}{x} \qquad a^b = e^{b\ln(a)} \qquad \ln(e^x) = x \qquad \ln(e) = 1 \qquad \ln(e^2) = 2 \qquad \ln(e^3) = 3$$

The graph of an inverse function is related to the graph of the function by reflection about the line $y = x$. In this next example we will plot several logarithmic functions and the related exponentials.

Maple Example Plot $\log_2(x)$, $\ln(x)$, $\log_{10}(x)$, their inverses 2^x, e^x, 10^x, and the line $y = x$.

```
> with(plots):
> A:=plot([log[2](x),ln(x),log[10](x)],x=(.3)..4,color=[red,green,blue]):
> B1:=plot(2^x,x=log[2](.3)..log[2](4),color=red):
> B2:=plot(exp(x),x=ln(.3)..ln(4),color=green):
> B3:=plot(10^x,x=log[10](.3)..log[10](4),color=blue):
> B4:=plot(x,x=log[2](.3)..4,color=khaki,scaling=constrained):
> display(A,B1,B2,B3,B4);
```



Maple Example: Plot $y = 3^x$ and its tangent line at $x = 0$ on the same coordinate axes.

Do you remember how we approximated the slope of a^x at $x = 0$ in **C1M3**? We defined a function

$$F := x \longrightarrow \frac{1}{2}\frac{a^x - a^{-x}}{x}$$

that calculated the slope of the line segment that joins the two points $P(-x, a^{-x})$ and $Q(x, a^x)$ spaced equally from $x = 0$. Then we selected different values of a and looked at values of this slope function as x got closer to 0. We are going to repeat this process and then compare our approximations with certain values.

```
> F:=x->(a^x-a^(-x))/(2*x);
```

$$F := x \longrightarrow \frac{1}{2}\frac{a^x - a^{-x}}{x}$$

```
> a:=2; seq(evalf(F(1/2^i),14),i=9..15);
```

$$a := 2$$

$$.69314739230, .69314723351, .6931471938, .6931471839, .6931471815, .6931471806, .693147181$$

```
> evalf(ln(2),14);
```

$$.69314718055995$$

```
> a:=3; seq(evalf(F(1/2^i),14),i=9..15);
```

$$a := 3$$

$$1.09861313170, 1.09861249940, 1.0986123413, 1.0986123019, 1.0986122923, 1.0986122897, 1.098612289$$

```
> evalf(ln(3),14);
```

$$1.0986122886681$$

```
> a:=10; seq(evalf(F(1/2^i),14),i=9..15);
```

$$a := 10$$

$$2.30259285469, 2.30258703341, 2.3025855781, 2.3025852143, 2.3025851234, 2.3025851002, 2.302585095$$

```
> evalf(ln(10),14);
```

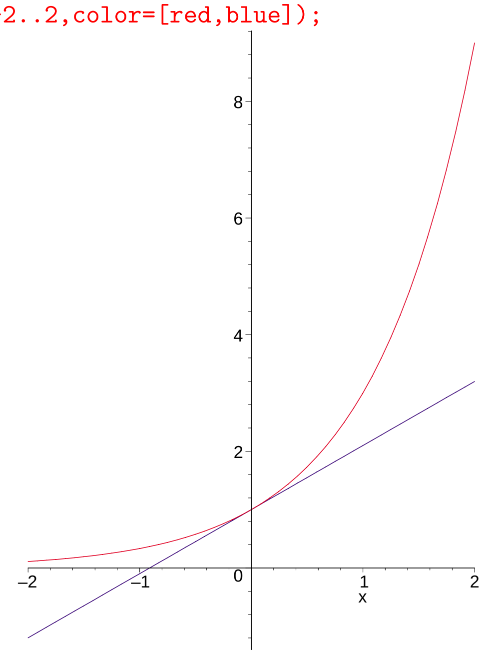
$$2.3025850929940$$

Although we have not proved it (yet), we are *strongly* suspicious that at $x = 0$ the slope of the line tangent to $y = a^x$ is $\ln(a)$. We will use this value for m in the equation for the line, $y - y_0 = m(x - x_0)$.

```
> eq1:=y-1=ln(3)*(x-0);
```

$$eq1 := y - 1 = \ln(3)\,x$$

```
> plot([3^x,ln(3)*x+1],x=-2..2,color=[red,blue]);
```



C1M4 Problems: Use Maple to display the following graphs:

1. $y = 5^x$ and $y = \log_5(x)$
2. $y = \log_{10}(x) + .01$ and $y = \dfrac{\ln(x)}{\ln(10)}$, $.2 \le x \le 5$ $\qquad \log_a(x) = \frac{\ln(x)}{\ln(a)}$
3. $y = \ln(2x) + .02$ and $y = \ln(x) + \ln(2)$, $.2 \le x \le 5$ $\qquad \ln(ab) = \ln(a) + \ln(b)$
4. $y = \ln(x^2) + .03$ and $y = 2\ln(x)$, $.2 \le x \le 5\pi$ $\qquad \ln(x^r) = r\,\ln(x)$
5. $y = 5^x$ and the line tangent at $x = 0$.

C1M5

Tangents and Velocity

Suppose that we operate an emergency vehicle on a busy highway and that the hospital that serves this community is on the highway 10 miles to the east of our base. An efficiency expert is hired to record our location electronically every minute for a week and it is determined that the hospital is located at $+10$ miles. The clock is started at midnight and at 12:03 AM a call from an accident two miles to the west comes in. We race to the scene, arriving at 12:06, put an injured patient in the vehicle, and head for the hospital 4 minutes later with a police escort, slowing somewhat for an intersection at the $+5$ mile mark. We reach the hospital at 12:19 AM.

Later, we are asked what our average speed was going to the hospital. Our efficiency expert looks at the data and points out that we went 12 miles in 9 minutes, so our average speed was $\dfrac{12}{9/60} = 80$ mph. Then we are asked if we ever exceeded 80 mph, which is our maximum emergency speed allowed by local statutes. Knowing that there is "ramp" time when we accelerate and decelerate, and that we had slowed to $50 - 60$ mph at that intersection, I concluded that we must have exceeded 80 mph at some time in the rush to the hospital. In fact, having glanced at the speedometer a couple of times, I knew that we had almost hit 90 mph once. After I admitted that we had exceeded 80 mph, the efficiency expert smiled knowingly and said,"At 12:12 you were at mile 1.6, and at 12:13 you were at mile 3.05, so your speed for that minute was $\dfrac{3.05 - 1.6}{1/60} = (1.45) \cdot 60 = 87$ mph."

I asked the expert if I could have the data and it was provided, so I decided to use Maple to analyze the situation. I began by listing the mile information and then I associated a time in minutes with each location, starting with 0. This generated a sequence, but by enclosing it in square brackets, it became a list. Then I plotted the data as a line graph and as a point graph, putting the plots on the same coordinate axes.

```
> restart:    with(plots):
> datapoints:=[0,0,0,0,-.3,-1.2,-2,-2,-2,-2,-1,.2,1.6,3.05,4.4,5.5,6.8,8.1,9.2,10];
```

$$datapoints := [0, 0, 0, 0, -.3, -1.2, -2, -2, -2, -2, -1, .2, 1.6, 3.05, 4.4, 5.5, 6.8, 8.1, 9.2, 10]$$

```
> nops(datapoints);
```

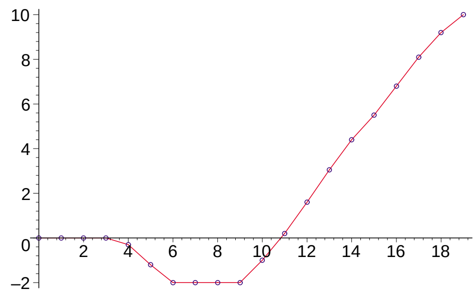
$$20$$

The command `nops` returns the number of parts of its argument. This told me that I had entered 20 values.

```
> seq1:=[seq([i,datapoints[i+1]],i=0..19)];
```

$$seq1 := [[0, 0], [1, 0], [2, 0], [3, 0], [4, -.3], [5, -1.2], [6, -2], [7, -2], [8, -2], [9, -2], [10, -1], [11, .2], [12, 1.6], [13, 3.05], [14, 4.4], [15, 5.5], [16, 6.8], [17, 8.1], [18, 9.2], [19, 10]]$$

```
> A:=pointplot(seq1,style=line,color=red,scaling=constrained):
> B:=pointplot(seq1,style=point,color=blue):
> display(A,B);
```



On the vertical axis we have the mile mark, and on the horizontal axis we have the time scale in minutes. Then I decided to compute the average velocity for each minute. By subtracting the old mile mark from the new one, I got the change in distance. Then I divided that by the change in time, which is $1/60^{\text{th}}$ of an hour, to get miles per hour.

```
> seq2:=seq((datapoints[i+1]-datapoints[i])/(1/60),i=1..19);
```

$$seq2 := 0, 0, 0, -18.0, -54.0, -48.0, 0, 0, 0, 60, 72.0, 84.0, 87.00, 81.00, 66.0, 78.0, 78.0, 66.0, 48.0$$

As I compared the speeds listed, I looked at the graph and noted that these numbers were just the slopes of the lines that joined the data points on the graph. Sure enough, our top speed listed was 87 mph and the slope of the line between 12 and 13 was the steepest of any of the slopes. I also realized that because we initially went west, the speeds then are shown as negative numbers. And, when we were stationary our speed was 0.

If the expert had been able to give data that was recorded every second, then it would have been overwhelming, but the speeds calculated would have been very accurate and the graph would have been almost smooth.

In this discussion the terms *velocity* and *speed* have been used as if they are interchangeable. This is misleading because they are related, but different. Think of speed as the absolute value of velocity. In our computations we determined the average velocity over a one minute time span. Speedometers record speed, not velocity, and they allow only for non-negative values. In addition to how fast the vehicle was moving, there was also a direction involved. Motion to the east produced a positive velocity, while motion towards the west yielded negative values. We chose east as our positive direction and recorded *signed values*, i.e. $+, -$ from the base, not just distance from that point. We did not really answer the question of how fast the vehicle was moving at some instant in time, rather we computed a sequence of average velocities over one minute intervals. As you should expect, in order to approximate the *instantaneous velocity* we would need to compute the average velocity over shorter and shorter time intervals. If this 'seems like *déjà vu* all over again', then you are right. Computing an instantaneous velocity and computing the slope of a tangent line are the same processes. Suppose that the dependent (y value) variable is distance. When the independent (x value) is distance we get a slope. When the independent variable is regarded as time (x value, but think of it as t), then our computation is a velocity.

What was our average velocity over the 19 minutes? The change in distance was 10 miles, while the change in time was 19/60 hours. This yields $10/(19/60)) = \frac{600}{19} \approx 31.6$ mph. There were probably two instants in time when our instantaneous velocity was $\frac{600}{19}$, once as we accelerated away from the accident and again as we decelerated as we approached the hospital. So this average value over a long time interval doesn't indicate our instantaneous velocity over this time period very well. To get a good approximation for an instantaneous velocity, we must use smaller and smaller time intervals. We are clearly dancing around a concept called a *limit*. In mathematics, whenever a limit is discussed there are always two ingredients involved, *accuracy* and *control*. In the section on limits we will be more precise, but in order to get a good approximation for our instantaneous velocity we 'invoke more control', i.e. we use smaller and smaller time intervals.

Slope

When calculating an approximation for the slope of the tangent line for $y = a^x$ at $x = 0$ we used a symmetric form of slope by taking a point on each side of 0. It might be helpful if you see that our approach is consistent with your prior experience. At x_0, you took an increment h which should be regarded as a small number, either positive or negative, but we almost always always draw the positive case. If $h > 0$ then on the x-axis the values $-h$ and h are $2h$ units apart. The usual form for the slope of a secant line of $y = f(x)$ at $x = x_0$ is given by

$$\frac{f(x_0+h)-f(x_0)}{h} \text{ on the right} \qquad \text{and} \qquad \frac{f(x_0-h)-f(x_0)}{-h} \text{ on the left}$$

and then we take the positive h smaller and smaller. Consider what happens when we insert 0 in the symmetric form in a clever way.

$$\begin{aligned}\frac{f(x_0+h)-f(x_0-h)}{2h} &= \frac{f(x_0+h)-f(x_0)+f(x_0)-f(x_0-h)}{2h} \\ &= \frac{1}{2}\left(\frac{f(x_0+h)-f(x_0)}{h} + \frac{f(x_0-h)-f(x)}{-h}\right)\end{aligned}$$

So using the symmetric form is really just taking the average of the right and left forms. When our function is smooth, a term that we will define more precisely later, we will get the same approximation either way. In fact, we will find our approximation by using only the form

$$\frac{f(x_0+h)-f(x_0)}{h}$$

and will permit h to be both positive and negative as it becomes small.

Maple Example: Use Maple to find an equation for the line tangent to $y = \sqrt{x}$ at $x = 2$. Plot the line and the function on the same coordinate axes.

```
> restart:    with(plots):
> f:=x->sqrt(x);  x0:=2;  y0:=f(x0);
```

$$f := \text{sqrt}$$
$$x0 := 2$$
$$y0 := \sqrt{2}$$

```
> F:=h->(f(x0+h)-f(x0))/h;
```

$$F := h \longrightarrow \frac{f(x0+h)-f(x0)}{h}$$

```
> S1:=seq(evalf(F(1/2^i)),i=6..12);
```

$$S1 := .35286555, .3532089, .3533810, .3534673, .353511, .353533, .353544$$

```
> S2:=seq(evalf(F(-1/2^i)),i=6..12);
```

$$S2 := .35424661, .3538993, .3537261, .3536395, .353596, .353574, .353563$$

```
> abs(S1[7]-S2[7]);
```

$$.000019$$

```
> m:=evalf(S1[7],4);
```

$$m := .3535$$

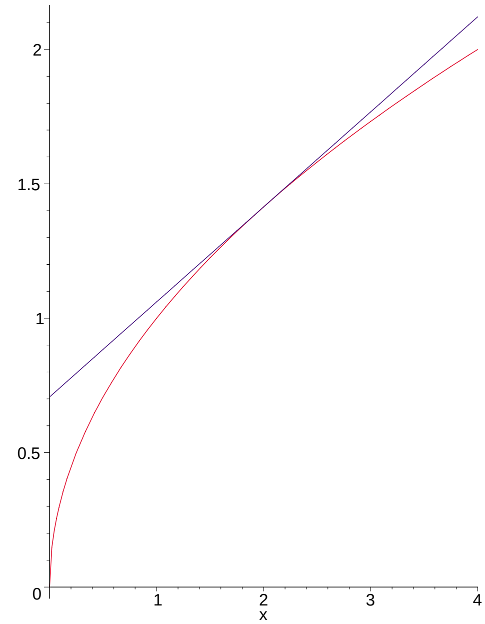
```
> eq1:=y-y0=m*(x-x0);
```

$$eq1 := y - \sqrt{(2)} = .3535 * x - .7070$$

```
> y:=solve(eq1,y);
```

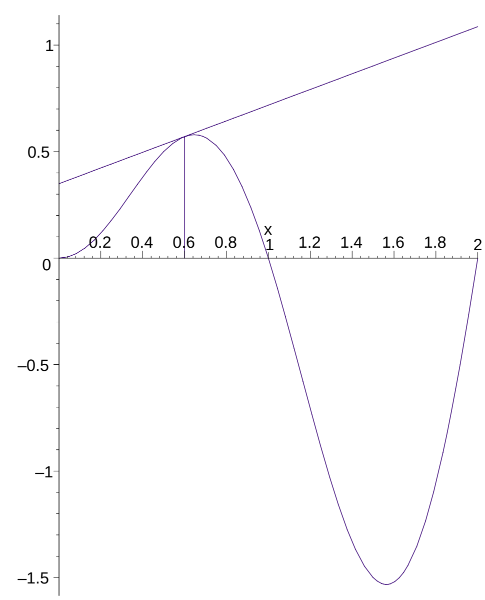
$$y := .7072135624 + .3535000000 * x$$

```
> P:=plot(y,x=0..4,color=blue):
> Q:=plot(f(x),x=0..4,color=red):
> display(P,Q);
```



Honesty requires that I show you an easy way to plot a function and its tangent line at a specified point using Maple. In the package `student` there is a command `showtangent` that does the job cleanly and without fuss. For example,

```
> restart:    with(student):
> showtangent(x*sin(Pi*x),x=.6,x=0..2,color=blue);
```



C1M5 Problems: Use Maple to solve the problems and plot the graphs.

1. Our efficiency expert from the text moved on to a different ambulance location and set up the same program. The data below was collected at one minute intervals and the first value is for 2 AM. The last value is at the hospital. Provide as much information about the ambulance run as possible, including the average velocities.

$$data = [0, 0, .3, 1.4, 2.7, 4, 4.8, 5.1, 5.1, 5.1, 5.1, 6, 7.3, 8.7, 10, 11, 11.7, 12.2, 12.6]$$

2. For $y = x\,\ln(x)$ and $x_0 = 3$, find an equation for the line tangent to the curve at x_0 and plot the graphs of the function and the line on the same coordinate axes.

3. For $y = x\,2^{-x}$ and $x_0 = \frac{-3}{2}$, find an equation for the line tangent to the curve at x_0 and plot the graphs of the function and the line on the same coordinate axes.

4. For $y = \sinh(x)$ and $x_0 = 2$, find an equation for the line tangent to the curve at x_0 and plot the graphs of the function and the line on the same coordinate axes.

Warning! It is perfectly reasonable to do Problem 2 and then, when doing 3 and 4, copy, paste it below, and change the values and execute. If you do, some values will carry over that you do not wish for Maple to remember. It is suggested that you insert a command line between the problems:

```
> restart:    with(plots):
```

C1M6

Limits

Graduating from high school in 1956 and driving cars with wide front seats and no seatbelts, one developed a sense of how things were going on a date by where she chose to sit. If she sat by the door and gripped the armrest with white knuckles, then things weren't going well. On the other hand, if she sat close, then one had to guard against grinning stupidly and spoiling the moment. "Close to" had real meaning. This is also true in mathematics whenever limits are being discussed. There are always two ingredients in the discussion and they behave as *accuracy* and *control*, and accuracy always precedes control. Let's consider the following sequence of numbers:

$$\begin{array}{ll}
x_1 = 1 & x_1^2 = 1 \\
x_2 = 1.4 & x_2^2 = 1.96 \\
x_3 = 1.41 & x_3^2 = 1.9881 \\
x_4 = 1.414 & x_4^2 = 1.999396 \\
x_5 = 1.4142 & x_5^2 = 1.99996164 \\
x_6 = 1.41421 & x_6^2 = 1.9999899241 \\
x_7 = 1.414213 & x_7^2 = 1.999998409369 \\
x_8 = 1.4142135 & x_8^2 = 1.99999982358225 \\
x_9 = 1.41421356 & x_9^2 = 1.9999999932878736 \\
x_{10} = 1.414213562 & x_{10}^2 = 1.999999998944727844 \\
. & . \\
. & .
\end{array}$$

The right column lists the squares of the left column, and we can see that the numbers in the right column are getting 'closer' to 2. Of course, this means that the numbers in the left column are getting 'closer' to $\sqrt{2}$. It is certainly fair to say that we are *approximating* $\sqrt{2}$ by increasing our accuracy one place each time we select a new number in the left column. Suppose that we wanted to approximate $\sqrt{2}$ to within .000001, which we may regard as an accuracy. Certainly if we go down our list to x_9 and look at $x_9^2 = 1.9999999932878736$ and at $|(x_9)^2 - 2| = .0000000067121264$, we see that we have achieved our accuracy. But, where does control fit in here? Note first that the numbers that we would list below x_9 would be even closer to $\sqrt{2}$ than x_9 is, we control the situation by selecting a point on our list where our accuracy is achieved for all the rest of the list. But, our focus will be on limits of functions rather than on sequences.

Let's assume that we have a laser attached to a rifle and that it is adjusted so that when the rifle is fired at a target, the bullet will strike exactly where the laser points at that instant. But, the person firing the weapon is not perfectly rigid and the aim can vary. Suppose further that the "bull's-eye" is ten inches across and we wish to hit the bull's-eye. In effect, we have selected an accuracy of 5 inches. To make things simpler, assume that the butt of the rifle is set on a fixed point and that we can measure the deviation of the barrel tip from 'perfect'. The question becomes, "What is a permissible deviation from perfect that ensures that the bullet will strike the bull's-eye?" This is our control. Suppose that when the deviation is less than .0357 inches from perfect, we are assured that the bull's-eye will be struck. For an accuracy of 5 inches we found a control of .0357 inches that would guarantee that the accuracy would be achieved. It would seem that for each accuracy selected, we could find a permissible deviation (control) that would guarantee that the bullet would strike the target within the chosen accuracy of the center of the bull's-eye. And this is how limits work.

Definition: (Limit of a function) We write

$$\lim_{x \to a} f(x) = L$$

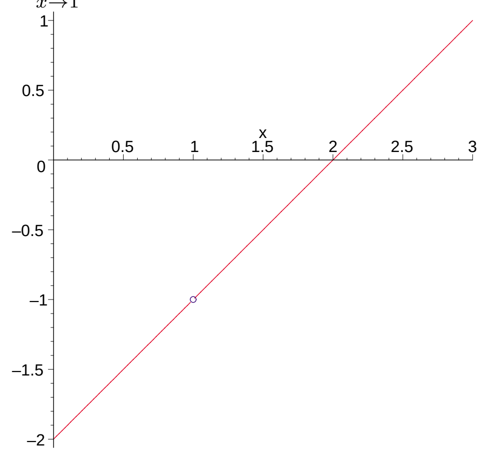
if for each $\epsilon > 0$ (accuracy) there is a number $\delta > 0$ (control), so that whenever $x \neq a$ and $|x - a| < \delta$, then it follows that $|f(x) - L| < \epsilon$.

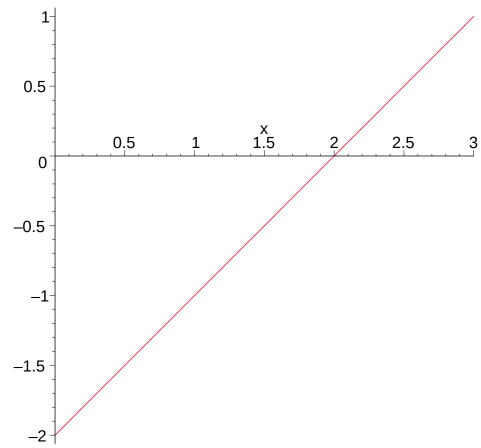
This is just a precise way of saying that the values of $f(x)$ are as close to L as we like whenever x is close enough to a, but is not equal to a. We call L 'the limit of $f(x)$ as x approaches a'.

Reminder: $|x - a| < \delta \iff a - \delta < x < a + \delta$

It is important now to comment on how when we are discussing a limit at $x = a$, we actually ignore the value, if any, of the function at a. In fact, we frequently look at only what is happening when $x < a$ (left-hand limit) or at when $x > a$ (right-hand limit). Then we compare our answers.

Suppose that $f(x) = \dfrac{x^2 - 3x + 2}{x - 1}$. We see immediately that f is not defined at $x = 1$. This does not mean that $\lim_{x \to 1} f(x)$ does not exist however. The graph is displayed on the left below.





The Maple code shown below produced the plot on the right above, in addition to the output that is displayed.

```
> restart:
> f:=x->(x^2-3*x+2)/(x-1);
```

$$f := x \to \frac{x^2 - 3x + 2}{x - 1}$$

```
> plot(f(x),x=0..3,scaling=constrained);
```

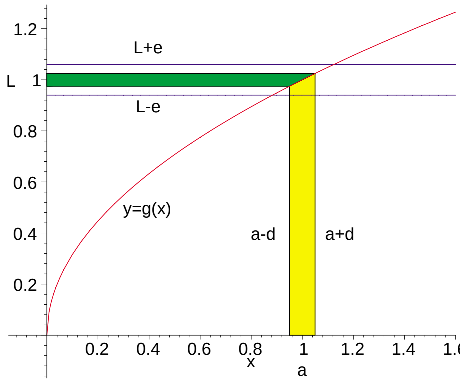
Output above on right

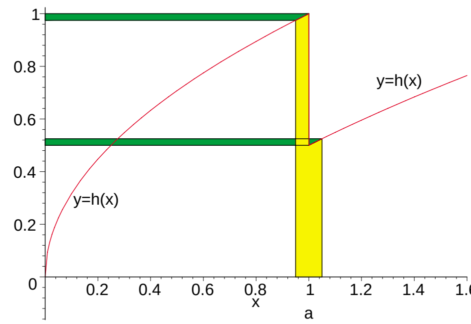
```
> Limit(f(x),x=1)=limit(f(x),x=1);
```

$$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1} = -1$$

What should we learn from this? Maple may not show a 'hole' in a graph, so we must not depend on Maple to show us problem points. When taking a limit in Maple and a capital 'L' is used, the expression is inert. That is, the operation is not executed. On the other hand, the lower case 'l' allows Maple to complete taking the limit.

Graphical discussion: The objective here is to show a picture of how a limit works by examining a graph of two functions, g and h, demonstrating the process. First, there is an "x" value a at which the limit is to be discussed. Then by some mysterious process we select the limit L and any accuracy $e > 0$. Now we draw two symmetric horizontal lines, $y = L - e$ and $y = L + e$ which determine our bounds. Then we consider two symmetric vertical lines about $x = a$ and all the points between, except a itself. The question is, "Can we move the lines together by making the number d smaller, but still positive, and have the functional values project up to the graph and over to the y-axis in such a way that the projected values remain between the two horizontal lines?" In the first diagram below the answer is "yes", and in the second the answer is "no". In fact, there is not even a candidate for L in the second one because of the manner in which the values of $h(x)$ are split.





In the case of $g(x)$ above, no matter how small we make e, we can always find a d small enough so that the diagram above on the left is valid. This means that $\lim\limits_{x \to a} g(x) = L$. The function $h(x)$, as drawn, has no limit at $x = a$.

We have mentioned one-sided limits, and now it is time to be specific about them. The function h above will serve as our mental picture. What happens when we restrict x so that $a - d < x < a$ (left-hand limit), or so that $a < x < a + d$ (right-hand limit)? As x approaches a from the left-hand side we sense that $h(x)$ is getting close to a number which we will call L_1. Also, as x approaches a from the right-hand side $h(x)$ is getting close to a number L_2. It is obvious that L_1 and L_2 are not the same numbers in this case. For completeness, we will provide precise definitions of one-sided limits.

Definition: (Left-hand limit of a function) We write

$$\lim_{x \to a^-} f(x) = L_1$$

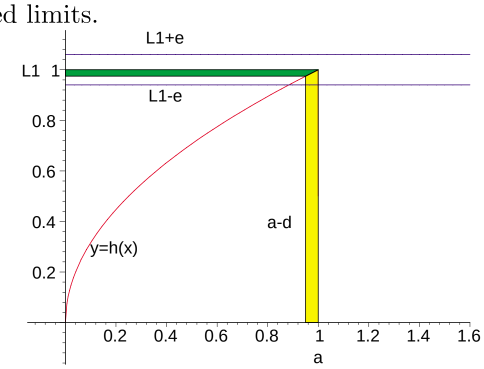
if for each $\epsilon > 0$ (accuracy) there is a number $\delta > 0$ (control), so that whenever $a - \delta < x < a$, then it follows that $|f(x) - L_1| < \epsilon$.

Definition: (Right-hand limit of a function) We write

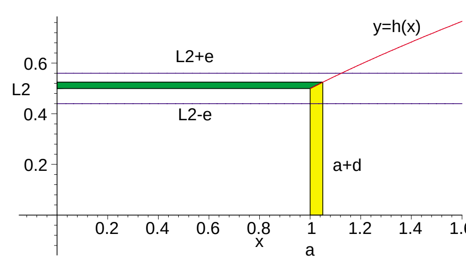
$$\lim_{x \to a^+} f(x) = L_2$$

if for each $\epsilon > 0$ (accuracy) there is a number $\delta > 0$ (control), so that whenever $a < x < a + \delta$, then it follows that $|f(x) - L_2| < \epsilon$.

Here is the graphical picture of what is happening. If we select a smaller accuracy ($e > 0$), then we may select a smaller control ($d > 0$), so that the functional values of those x's lie within the requested range close to the limit. Note that we only provided one half of the graph of $h(x)$ in each case. This is to emphasize that only what happens on the left and right sides respectively of $x = a$ matters when determining the one-sided limits.



$$\lim_{x \to a^-} h(x) = L_1$$



$$\lim_{x \to a^+} h(x) = L_2$$

It should be obvious that there is a relationship between limits and one-sided limits. We usually tie this up nicely with a theorem that shows the equivalence of two statements. But, it is a little easier for the student to understand if it is stated as two related theorems and a corollary, and that is what we will do here.

Theorem A:

$$\lim_{x\to a} f(x) = L \quad \Longrightarrow \quad \lim_{x\to a^-} f(x) = L \ \textbf{and} \ \lim_{x\to a^+} f(x) = L$$

Theorem B:

$$\lim_{x\to a^-} f(x) = L_1 \text{ and } \lim_{x\to a^+} f(x) = L_2 \ \textbf{and} \ L_1 = L_2 \quad \Longrightarrow \quad \lim_{x\to a} f(x) = L_1 \ (= L_2)$$

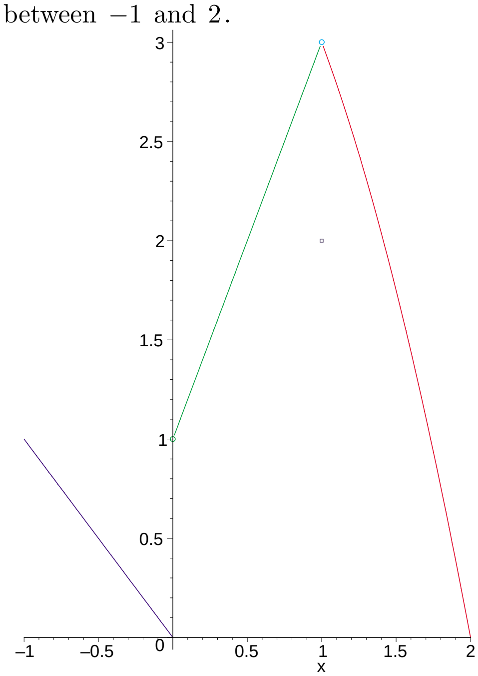
Corollary:

$$\lim_{x\to a^-} f(x) = L_1 \text{ and } \lim_{x\to a^+} f(x) = L_2 \ \textbf{and} \ L_1 \neq L_2 \quad \Longrightarrow \quad \lim_{x\to a} f(x) \ \textbf{does NOT exist}$$

Maple Example: A function f may be defined by

$$f(x) = \begin{cases} -x, & \text{if } -1 \le x \le 0; \\ 2x+1, & \text{if } 0 < x < 1; \\ 2, & \text{if } x = 1; \\ 4 - x^2, & \text{if } 1 < x \le 2. \end{cases}$$

Here is a look at the graph of f between -1 and 2.



Suppose we want to find if f has a limit at $x = 0$ and $x = 1$. We look at the graph for inspiration and realize that if we stay to the left of $x = 0$, then the function is tending to 0 as x gets closer to the value 0. Also, if we stay to the right of $x = 0$ and allow x to decrease towards 0, then the values of f are getting close to 1. It would seem that the left-hand limit is 0 and the right-hand limit is 1. Then, if we take the same approach at $x = 1$, but we must ignore $x = 1$, and allow x to get closer and closer to 1, then $f(x)$ will get closer to 3, no matter which side of 1 x happens to lie on. All this suggests intuitively that

$$\lim_{x\to 0^-} f(x) = 0, \ \lim_{x\to 0^+} f(x) = 1 \implies \lim_{x\to 0} f(x) \text{ does not exist}$$

$$\lim_{x\to 1} f(x) = 3$$

Now we turn to Maple.

```
> restart:
> f:=x->piecewise(x>=-1 and x<=0,-x,x<1,2*x+1,x=1,2,x>1 and x<=2,4-x^2);
```

$$f := x \to \mathit{piecewise}(-1 \le x \ \textbf{and} \ x \le 0, -x, x < 1, 2x+1, x = 1, 2, 1 < x \ \textbf{and} \ x \le 2, 4 - x^2);$$

```
> limit(f(x),x=0,left);
```

$$0$$

```
> limit(f(x),x=0,right);
```

$$1$$

```
> limit(f(x),x=0);
```

$$undefined$$

```
> limit(f(x),x=1);
```

$$3$$

```
> f(-1/2); f(1/2);  f(1);
```

$$\frac{1}{2}$$

$$2$$

$$2$$

You see above an example of how to define a function piecewise. You list the conditions on x first, and then the function's values for those x's. When there are two conditions on x, you must separate them with 'and'. You see how easy it is to obtain regular and one-sided limits.

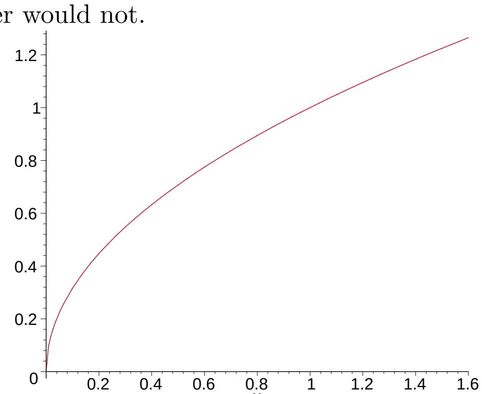
C1M6 Problems: Use Maple to plot the graphs and to find the limits at the indicated points, if they exist.

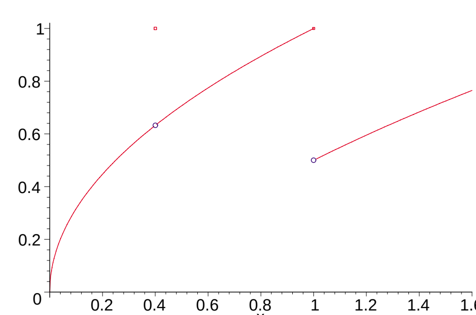
1. $f(x) = \dfrac{\sin(2x)}{3x}$, $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$, at $x = 0$.
2. $g(x) = \sin\left(\dfrac{1}{x}\right)$, $-\pi \le x \le \pi$, at $x = 0$.
3. $h(x) = x\,\sin\left(\dfrac{2}{x}\right)$, $-\pi \le x \le \pi$, at $x = 0$.
4. $F(x) = \dfrac{\sin(x) - x}{x^3}$, $-\pi \le x \le \pi$, at $x = 0$.
5. $G(x) = \begin{cases} x+1, & \text{if } -1 \le x \le 0; \\ x^2, & \text{if } 0 < x \le 2; \\ 6-x, & \text{if } 2 \le x \le 4. \end{cases}$, at $x = 0$ and $x = 2$.

C1M7

Continuity

To paraphrase a judge who said something like "Pornography might be hard to define, but I know it when I see it.", a similar statement about continuity might be, "Continuity might be hard to define, but I know it when I see it." In the two diagrams which follow, when you look at them you will have an immediate sense that one function is continuous and the other is discontinuous at two different points. You might think of the graph of the function as a wire and one graph would allow an electric current to flow through it while the other would not.





As it turns out, we may define *continuity of a function at a number a* fairly easily, but the application and understanding of the concept might take some effort.

Definition: (Continuity of a function at a number a) A function f is **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a)$$

This says two things: (1) the limit of f at $x = a$ exists; and (2) the limit has the value $f(a)$.

We repeat the definition of limit from a previous module and follow it with a definition of continuity that is equivalent to the one above.

Definition: (Limit of a function) We write

$$\lim_{x \to a} f(x) = L$$

if for each $\epsilon > 0$ (accuracy) there is a number $\delta > 0$ (control), so that whenever $x \neq a$ and $|x - a| < \delta$, then it follows that $|f(x) - L| < \epsilon$.

Definition: (Continuity of a function at a number a) We say that a function f is continuous at a number a if for each $\epsilon > 0$ (accuracy) there is a number $\delta > 0$ (control), so that whenever $|x - a| < \delta$, then it follows that $|f(x) - f(a)| < \epsilon$.

This is just a precise way of saying that the values of $f(x)$ are as close to $f(a)$ as we like, whenever x is close enough to a. Here we do not exclude $x = a$ as we would when discussing the limit at a.

"What would this look like on a graph?", you may well ask. It will look just like a limit illustration, except that one need not exclude $x = a$ from the discussion.

As you might expect, we may discuss continuity from one side or the other. If the limit taken from the left at a differs from the limit taken from the right, then the limit does not exist at a so continuity there is not possible. But, if the left-hand limit is $f(a)$, then f is *continuous from the left* at a. A similar statement about the right-hand limit is valid. This leads to a simple statement which we will list as a theorem:

Theorem: A function f is continuous at $x = a$ if and only if f is continuous from the left at a and f is continuous from the right at a.

It is not fashionable in many calculus courses to actually *prove* that a function is continuous. And, this author agrees that this should not be over-stressed. But, it is instructive to see *how* this is done in some simple case at least once, because it reinforces how accuracy and control are involved. So that we may not be accused of focusing on theory we take a

TIME OUT!

Let's show that the function $f(x) = mx + b$ is continuous at some value a for the case where $m \neq 0$. The main problem here is that at this point we wouldn't know when we have accomplished our task. So, let's begin with what we would like to end up with. We would see something like

$$|x - a| < \delta \quad \Longrightarrow \quad |(mx + b) - (ma + b)| < \epsilon$$

where we were given $\epsilon > 0$ and found a $\delta > 0$ that made this statement valid. So, we go to a piece of scrap paper for some figuring.

SCRAP PAPER:

Now we can try and work this backwards.

$$\begin{aligned} |(mx + b) - (ma + b)| &< \epsilon \\ |(mx - ma)| &< \epsilon \\ |x - a| &< \frac{\epsilon}{|m|} \qquad \text{permitted since } m \neq 0 \end{aligned}$$

Aha! This last inequality looks remarkably like $|x - a| < \delta$. Now we are ready to work forward because each of our inequalities on our scrap paper were reversible.

Back to the good paper!

Suppose that we are given $\epsilon > 0$, which is our requested accuracy. We choose $\delta = \dfrac{\epsilon}{|m|}$, which is our control. Then, for each x so that $|x - a| < \delta$ we may write

$$\begin{aligned} |x - a| &< \delta = \frac{\epsilon}{|m|} \\ |m|\,|x - a| &< \epsilon \\ |mx - ma| &< \epsilon \\ |(mx + b) - (ma + b)| &< \epsilon \end{aligned}$$

This means that whenever $|x-a|<\delta$, it follows that $|(mx+b)-(ma+b)|<\epsilon$, W^5*. We have proved that straight line functions are continuous at every point. Maybe the **P**roof **C**ops won't catch us.

TIME IN!

In your text you should find a theorem or statement about building continuous functions from other continuous functions. Suppose that c is a constant and f and g are continuous functions whose domains and ranges align so that the following functions make sense. Then

$$c \qquad f+g \qquad f-g \qquad fg \qquad cf \qquad \frac{f}{g} \qquad f\circ g$$

are all continuous. It is easy to see why constant functions are continuous. They only take on one value, c, so $|f(x)-c|=0<\epsilon$ for every $\epsilon>0$ and all x. We know that functions of the form $f(x)=mx+b$ are continuous, so by applying all this we may conclude that:

Theorem: (Polynomials) If $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots a_2x^2+a_1x+a_0$ for constants $a_0,a_1,\ldots,a_n$, then f is a continuous function at every point.

Although we won't state it as a theorem, all trigonometric, exponential and logarithmic functions are continuous at *every point of their domain.* Once we accept all this, we know that a function like

$$g(x)=3\,e^{2x}\sin^2(3x^2+5)+\ln(x^2+1)$$

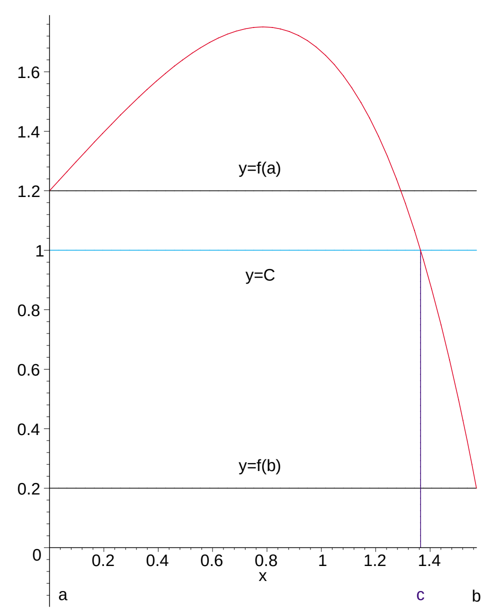
is continuous everywhere. We have used generously the concept from above that verbalizes as, "A continuous function of a continuous function is itself continuous." Let's state this as a theorem.

Theorem: (Composition of Continuous Functions) If g is continuous at a and f is continuous at $g(a)$, then $f\circ g$ is continuous at a, where $(f\circ g)(x)=f\,(g(x))$.

The concept of continuity is a powerful one. In mathematics, whenever the conclusion of a theorem includes a phrase like "there exists a ... ' you have a potentially strong and useful theorem. The reason for this is because it essentially says "there is a solution to ..." and knowing that a problem has a solution means that your efforts to find a solution are not doomed from the start. The theorem which follows is one of two such theorems involving continuity on a closed interval that we will state here. Recall that when we say that f is continuous on $[a,b]$, at $x=a$ we mean *continuous from the right* and at $x=b$ we mean *continuous from the left.* To keep this straight, remember that you work from the inside of the interval towards the endpoint.

Intermediate Value Theorem: Suppose that f is a continuous function on the closed interval $[a,b]$ and that C is a value strictly between $f(a)$ and $f(b)$ (either $f(a)<C<f(b)$ or $f(b)<C<f(a)$). Then **there exists** a number c so that $f(c)=C$.

This says that if you have two distinct values on the y-axis and $f(a)$ is one of them and $f(b)$ is the other, and if you draw a horizontal line between them, then the line **must** meet the graph at some point between a and b. There may be many such points, but you are *guaranteed at least one.* The obvious diagram follows.



* **W**hich **W**as **W**hat **W**e **W**anted

Maple Example: Suppose that $f(x) = e^x \cos(x) + (.2)$. For $C = 1$, illustrate the Intermediate Value Theorem.

```
> restart:
> f:=x->exp(x)*cos(x)+.2;
```

$$f := x \to e^x \cos(x) + .2$$

```
> f(0);  f(Pi/2);  pi/2=evalf(Pi/2);
```

$$1.2$$
$$.2$$
$$\frac{1}{2}\pi = 1.570796327$$

```
> C:=1
```

$$C := 1$$

```
> c:=fsolve (f(x)=C,x,0..Pi/2);
```

$$c := 1.365054488$$

The diagram above is actually this example. If you try to use `solve` you do not get a solution between 0 and $\pi/2$. However, `fsolve` provides a floating point answer and it allows you to specify a range from which to seek a solution. Note the omission of `x=` before the range, which is different from most other cases.

Application of the Intermediate Value Theorem: (optional) Choose any great circle of Earth, C, such as the Equator. Suppose that at some instant in time we are able to record the temperature at every point on the circle. Let R be the radius of Earth. Because $P(t) = [R\cos(t), R\sin(t)]$ traces out a circle of radius R for $0 \le t \le 2\pi$, we could (with a little effort) define a function F so that $F(t)$ would produce the temperature of the point $P(t)$ on the circle C for $0 \le t \le 2\pi$. Because there will be no jumps in temperature as we make small changes in our position on C, we see that F will be a continuous function of t on the closed interval $[0, 2\pi]$. Note that $P(0) = P(2\pi)$ because 0 and 2π represent the same point on Earth. As a result, $F(0) = F(2\pi)$. We will show why there must be two points on the circle which are antipodal and have the same temperature. Points are *antipodal* if they lie at opposite ends of a diameter. Here this will mean that $F(t_0) = F(t_0 + \pi)$ for some value t_0 of t between 0 and π. Let's show why this is true.

Begin by defining a function G as $G(t) = F(t) - F(t + \pi)$. As the composition of two continuous functions, $F(t + \pi)$ must be continuous. This means that G is the difference of two continuous functions, so it is also continuous. We seek a solution to the equation $G(t) = 0$. Let's see what happens at 0. Since $G(0) = F(0) - F(\pi)$ there are two possibilities. First, if $G(0) = 0$ we see that $F(0) = F(\pi)$ and we have found our solution. If $G(0) \ne 0$ then $G(0)$ and $G(\pi) = F(\pi) - F(2\pi) = F(\pi) - F(0)$ are opposite in sign. One is positive and the other is negative. By the Intermediate Value Theorem there must be a point t_0 between 0 and π for which $G(t_0) = 0$ or $F(t_0) = F(t_0 + \pi)$. W^5

Let's show an example using Maple.

```
> restart:   with(plots):   with(plottools):
> F:=t->t*(2*Pi-t)*(sin(sqrt(2)*t))^2/5+.5;
```

$$F := t \to \frac{1}{5}t(2\pi - t)\sin(\sqrt{2}t)^2 + .5$$

```
> F(0);  F(2*Pi);
```

$$.5$$
$$.5$$

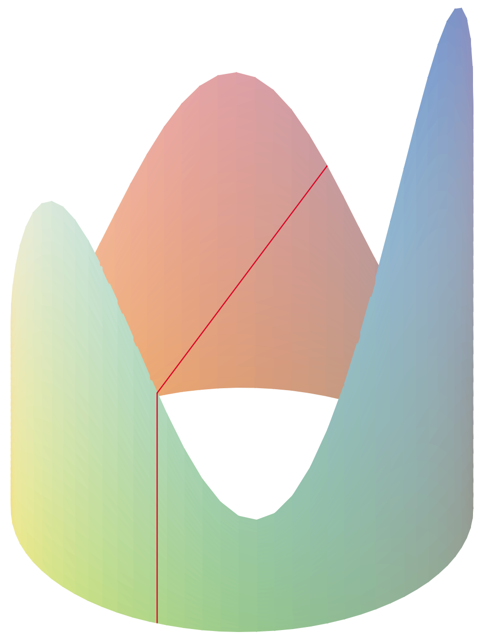
```
> a:=fsolve(F(t)=F(t+Pi),t,0..Pi);
```

$$a := 1.786163402$$

```
> evalf(F(a));   evalf(F(a+Pi));
```

$$1.035640467$$
$$1.035640466$$

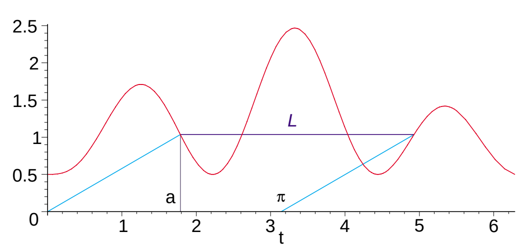
```
> P:=plot3d([cos(t),sin(t),z],t=0..2*Pi,z=0..F(t),grid=[80,30],
> orientation=[124,66],style=PATCHNOGRID):
> L:=line([cos(a),sin(a),F(a)],[cos(a+Pi),sin(a+Pi),F(a+Pi)],color=red,thickness=2):
> L1:=line([cos(a),sin(a),0],[cos(a),sin(a),F(a)],color=red,thickness=2):
> L2:=line([cos(a+Pi),sin(a+Pi),0],[cos(a+Pi),sin(a+Pi),F(a+Pi)],color=red,thickness=2):
> display(P,L,L1,L2);
```



DON'T PANIC! Three-dimensional plotting is usually found in Calculus III, or at least in the last part of Calculus II. Since this example is optional, I thought I would toss in the picture just for fun. I tried to color the graph so that it was red when the temperature was high and blue when it was cold, but that seemed to require more trouble than it was worth. Hopefully, you will note that the line across the figure is horizontal, illustrating that these antipodal points have the same temperature. No effort was made to scale the function so that the values were plausible temperatures.

Continuing with this same example, let's look at what is happening two-dimensionally. Because $F(0) = F(2\pi)$, there must be a point a so that $F(a) = F(a+\pi)$, as we said earlier. What this means is, if we take a line, L, that is of length π and hold it horizontally and slide the left end along the curve, then there is a value a where the right end will lie on the curve. Here is the picture and the Maple that produced it as we continue in the worksheet.

```
> P2:=plot(F(t),t=0..2*Pi,scaling=constrained):
> M1:=line([a,F(a)],[a+Pi,F(a+Pi)],color=blue,thickness=2):
> M2:=line([0,0],[a,F(a)],color=cyan,thickness=2):
> M3:=line([Pi,0],[a+Pi,F(a+Pi)],color=cyan,thickness=2):
> M4:=line([a,0],[a,F(a)],color=navy):
> V1:=textplot([a-.13,.19,"a"]):
> V2:=textplot([Pi,.19,"p"],font=[SYMBOL,12]):
> V3:=textplot([3.3,f(a)+.2,"L"],font=[HELVETICA,OBLIQUE,14],color=blue)
> display(P2,M1,M2,M3,M4,V1,V2,V3);
```



The other big theorem for continuous functions on closed intervals involves maximization and minimization. Before we discuss this problem, let's look at a quick example of a situation where there is no solution to our question.

Example: Suppose that $f(x) = x$ on the *open* interval $(0,1)$. Then, there is no point x_0 in this interval such that $f(x_0) \leq f(x)$ for all x in the interval. Similarly, there is no point x_1 in $(0,1)$ for which $f(x) \leq f(x_1)$ for all x in this interval. Suppose we think that x_0 satisfies the condition for being a minimum. Then $\hat{x} = \frac{x_0}{2} < x_0$ violates this condition because $f(\hat{x}) = \frac{x_0}{2} < x_0 = f(x_0)$. Since $(0,1)$ is not a *closed* interval, there are no points which serve as a maximum or a minimum for f. Certainly, f is continuous on the open interval.

Extreme Value Theorem: If f is a continuous function of a closed interval $[a,b]$, then there are numbers c and d in $[a,b]$ so that $f(c) \le f(x) \le f(d)$ for all x in $[a,b]$.

We say that c is a *minimum point* and d is a *maximum point* for f on $[a,b]$, while $f(c)$ and $f(d)$ are the (global) *minimum* and (global) *maximum* of f respectively.

Later you will deal with these concepts in more depth and how to locate c and d. For now, it suffices to say that the power of continuity and dealing with a closed interval guarantees a solution to finding a maximum or a minimum point.

Maple Example: Find the maximum value attained, the minimum value attained, and points where these values are achieved for $g(x) = e^{\sin(x)}$ on $[0, 2\pi]$.

Exponential functions are continuous at all points, as are $\sin(x)$ and $\cos(x)$. Other trigonometric functions are continuous at *all points of their domains.* So, our function, g, is continuous as the composition of two continuous functions. Our interval is closed, so the Extreme Value Theorem guarantees that the extrema are attained on the closed interval $[0, 2\pi]$. We will find the answers and then plot the graph and show you all the gory details of that plot.

```
> restart:    with(plots):    with(plottools):
> g:=x->exp(sin(x));
```

$$g := x \to \mathbf{e}^{\sin(x)}$$

```
> C:=minimize(g(x),x=0..2*Pi,location);
```

$$C := \mathbf{e}^{(-1)}, \{\left[x = \frac{3}{2}\pi, \mathbf{e}^{(-1)}\right]\}$$

Because we added `location` to the last command, our answer is in two parts. So, when we want to refer to the minimum value we must use `C[1]` instead of `C` because that value is the first thing listed in the output. Similarly with `maximize`. In Maple, `D` has special meaning, so it cannot be used as a name for a variable. We use `D1` instead.

```
> c:=fsolve(g(x)=C[1],x,0..2*Pi);
```

$$c := 4.712388980$$

```
> evalf(3*Pi/2);
```

$$4.712388981$$

```
> D1:=maximize(g(x),x=0..2*Pi,location);
```

$$D1 := \mathbf{e}, \{\left[x = \frac{1}{2}\pi, \mathbf{e}\right]\}$$

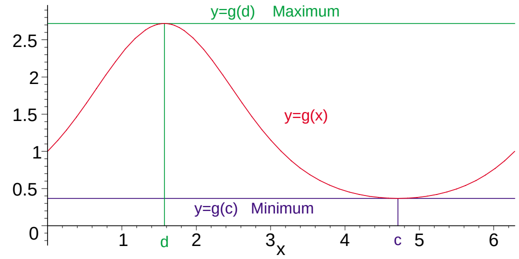
```
> d:=fsolve(g(x)=D1[1],x,0..2*Pi);
```

$$d := 1.570796327$$

```
> evalf(Pi/2);
```

$$1.570796327$$

```
> H:=plot(g(x),x=0..2*Pi,color=red,thickness=2):
> L5:=line([c,0],[c,g(c)],thickness=2,color=blue):
> L6:=line([d,0],[d,g(d)],thickness=2,color=green):
> L7:=line([0,g(d)],[2*Pi,g(d)],thickness=2,color=green):
> L8:=line([0,g(c)],[2*Pi,g(c)],thickness=2,color=blue):
> C1:=textplot([Pi,2.9,"y=g(d)    Maximum"],font=[HELVETICA,12],color=green):
> C2:=textplot([2.85,.25,"y=g(c)   Minimum"],font=[HELVETICA,12],color=blue):
> C3:=textplot([c,-.2,"c"],font=[HELVETICA,12],color=blue):
> C4:=textplot([d,-.2,"d"],font=[HELVETICA,12],color=green):
> C5:=textplot([3.5,1.5,"y=g(x)"],font=[HELVETICA,12],color=red):
> display(H,L5,L6,L7,L8,C1,C2,C3,C4,C5);
```



We activated the package `plottools` because it contains the `line` command which we needed to connect the points. We could have plotted the horizontal lines as functions, but cutting and pasting seemed a little easier. The `minimize` and the `maximize` commands require an expression and accept an optional range. Another convenient option is to include `location` and actually identify where the extrema occur.

We have provided a lot more Maple detail than is really necessary. On the other hand, if you never see "how" to do something, then you may never improve your skills.

C1M7 Problems: Use Maple to plot the graphs and to find the requested values.

1. If $f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$, use the limit definition to see if f is continuous at:
(a) $x = 0$;
(b) $x = \frac{6}{\pi}$.

2. If $g(x) = \begin{cases} x\sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$, use the limit definition to see if g is continuous at:
(a) $x = 0$;
(b) $x = \frac{6}{\pi}$.

3. For $K(t) = e^t \sin(3t)$ on $[0, \pi]$:
(a) determine if there **must** be a zero for K between 1.5 and 2.5, and find it if there must be;
(b) determine if there **must** be a solution to $K(t) = K(t + \pi/2)$ on $[0, \pi/2]$, and find it if there must be;
(c) find the maximum and minimum values of K and where they are located.

4. For $G(t) = t(2 - t)\sin(\sqrt{5}t) + \frac{1}{2}$ on $[0, 2]$:
(a) determine if there **must** be a point between 0 and 1 for which $G(t) = 1$, and find it if there must be;
(b) determine if there **must** be a solution to $G(t) = G(t + 1)$ on $[0, 1]$, and find it if there must be.
(c) find the maximum and minimum values of G and where they are located.

C1M8

Limits Involving Infinity

One way of categorizing the involvement of infinity in this discussion is to refer to *unbounded domains* and to *unbounded functions*. Let's deal with unbounded domains first. The basic question here is to see if the values of a function, $f(x)$, are getting close to a single value as we let x get larger and larger without bound. It will be a lot simpler if we recall that *accuracy* and *control* play a role here and then sort out how that happens. The role of accuracy is just like earlier cases, so the conclusion will look like

$$|f(x) - L| < \epsilon$$

whenever we invoke the control. We want this to hold for all "large" x. Our control here occurs as a number, $M > 0$, that we intuitively think of as large. We invoke our control by saying that x must satisfy

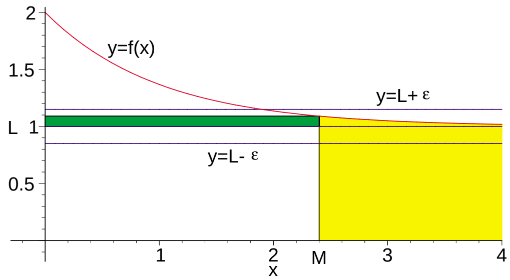
$$x > M \qquad \text{or} \qquad x \geq M$$

Definition: (Limit of a function at ∞) We write

$$\lim_{x\to\infty} f(x) = L$$

if for each $\epsilon > 0$ (accuracy) there is a number $M > 0$ (control), so that whenever $x > M$, then it follows that $|f(x) - L| < \epsilon$.

This is saying geometrically that whenever we draw two horizontal lines, $y = L + \epsilon$ and $y = L - \epsilon$, then from some point ($x = M$) on, the graph will lie between the lines.

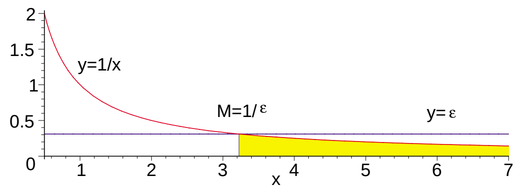


One of the first things that we must do is discuss $1/x$ as x gets large.

Example: Let's establish this basic fact:

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

If we are given an accuracy $\epsilon > 0$, then we choose $M = \frac{1}{\epsilon}$ as our control. It isn't too difficult to see that $M < x \implies \frac{1}{x} < \frac{1}{M} = \epsilon$.



Maple Example: Evaluate $\sqrt{x^2 + 6x + 4} - x$.

```
> restart:
> limit(sqrt(x^2+6*x+4)-x,x=infinity);
```

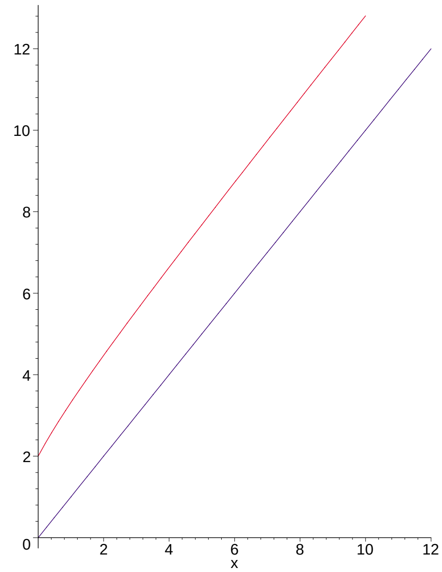
$$3$$

Whoa! You might say that this result is not intuitively obvious. Let's see if we can understand what happened. Follow the development algebraically and then we will take the limit at the end.

$$\begin{aligned}
\sqrt{x^2+6x+4} - x &= \left(\sqrt{x^2+6x+4} - x\right) \cdot \left(\frac{\sqrt{x^2+6x+4}+x}{\sqrt{x^2+6x+4}+x}\right) \\
&= \frac{x^2+6x+4-x^2}{\sqrt{x^2+6x+4}+x} \\
&= \frac{6x+4}{\sqrt{x^2+6x+4}+x} \\
&= \frac{6+\frac{4}{x}}{\frac{\sqrt{x^2+6x+4}+x}{x}} \\
&= \frac{6+\frac{4}{x}}{\sqrt{\frac{x^2+6x+4}{x^2}}+1} \\
&= \frac{6+\frac{4}{x}}{\sqrt{1+\frac{6}{x}+\frac{1}{x^2}}+1} \longrightarrow \frac{6}{2} = 3
\end{aligned}$$

The limit is taken as x goes to infinity, causing several terms to go to 0. Let's consider the graphs of $\sqrt{x^2+6x+4}$ and x. You will see that the difference does tend to 3.

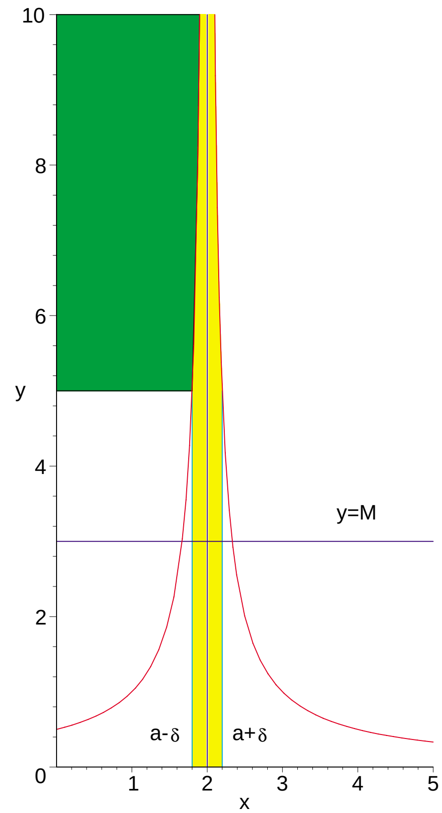
```
> H1:=plot(sqrt(x^2+6*x+4),x=0..10):
> H2:=plot(x,x=0..12,color=blue):
> display(H1,H2);
```



Now let's turn our attention to functions which are unbounded as x approaches a given value. Here our accuracy is given as a 'large' number M and we use the usual control, $\delta > 0$ for x.
Definition: (Limit of an unbounded function) We write

$$\lim_{x \to a} f(x) = \infty$$

if for each $M > 0$ (accuracy) there is a number $\delta > 0$ (control), so that whenever $x \neq a$ and $|x - a| < \delta$, then it follows that $M < f(x)$.



We must be very careful here. Although we write $\lim_{x \to a} f(x) = \infty$, in fact, **THIS LIMIT DOES NOT EXIST**. But, saying that the limit is infinity does provide us with useful information. It tells us that as x gets close to a, the values of $f(x)$ are getting larger and larger without any bound on these values.

In the diagram above, as x approaches a we see that $f(x)$ gets large, no matter which side we approach from. In the next example we change that so that we must consider $-\infty$ as a 'limit'.

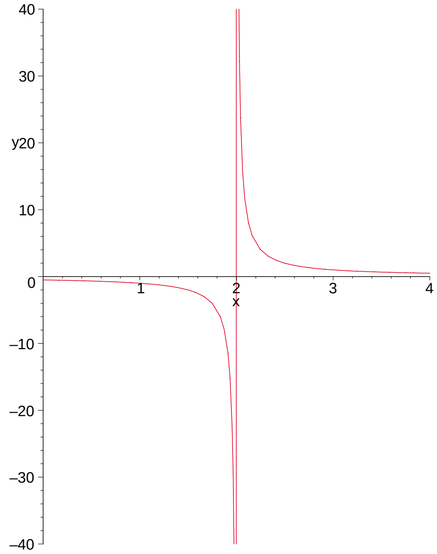
Example: Discuss $\lim\limits_{x\to 2} \dfrac{1}{x-2}$.

This discussion is going to be intuitive rather than precise. Let's look at the graph of $y = \frac{1}{x-2}$ first.

```
> f:=x->1/(x-2);
```

$$f := x \to \frac{1}{x-2}$$

```
> plot(f(x),x=0..4,y=-40..40);
```



Note that we inserted a range for y here so that we have control over the scale. There is a vertical asymptote at $x = 2$ and you see that Maple deals with the jump from large negative to large positive values by drawing the vertical line that is not really part of the graph. It is clear that as x approaches 2 from the left the values are getting larger and larger negatively. From the right, values are getting large without bound. Let's allow Maple to sum things up.

```
> limit(f(x),x=2,left);
```

$$-\infty$$

```
> limit(f(x),x=2,right);
```

$$\infty$$

```
> limit(f(x),x=2);
```

$$undefined$$

C1M8 Problems: Use Maple to plot the graphs and to discuss the limits at the specified points. Where necessary, insert a vertical range into the plot. For situations like the last example, use one-sided limits.

WARNING! Spelling is important. Maple recognizes `infinity`, but it ignores creative variations.

1. For $f(x) = \tan(x/2)$, discuss $\lim\limits_{x\to\pi} f(x)$.

2. Discuss: (a) $\lim_{x\to 0^+} \ln(x)$ and (b) $\lim_{x\to 0^+} x\,\ln(x)$.

3. Just evaluate: (a) $\lim\limits_{x\to\infty}(1+1/x)^x$; (b) $\lim\limits_{x\to\infty}(1+\sin(\pi/(2x))^x$

4. Given: $f(x) = \dfrac{x^{50}}{e^x}$ Find the exact and floating point values:

   (a) $f(3)$; (b) $f(20)$; (c) $f(50)$; (d) $f(300)$; (e) $\lim\limits_{x\to\infty} f(x)$

C1M9

The Derivative of a Function

Suppose that we have a function f whose domain includes an open interval containing the value a.

Then we define the *derivative of f at a*, $f'(a)$, by

$$\begin{aligned} f'(a) &\equiv \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \end{aligned}$$

Noting the relationship $x = a + h$ or $x - a = h$, we see that the two limits above are the same. We will use either format without remark.

It is implicit from our definition that we obtain a new function from f, namely f', when we differentiate (take the derivative of) f. We might have a function whose value is $\sin(ax)$. Well, what is the variable here? a? x? For that reason, when we use Maple to evaluate derivatives we must identify the variable with respect to which we are differentiating.

```
> restart:
> f:=x->x*sin(x);
```

$$f := x \to x\,\sin(x)$$

```
> A:=diff(f(x),x);
```

$$A := \sin(x) + x\,\cos(x)$$

```
> fprime:=unapply(A,x);
```

$$\mathit{fprime} := x \to \sin(x) + x\,\cos(x)$$

You see how to differentiate a function (actually, $f(x)$ is an expression) with respect to x and may note how the output is an expression. We convert the output to a function by using `unapply`, again identifying the variable to be used. Let's see how Maple handles the process if we use the two formats in our definition.

```
> limit((f(a+h)-f(a))/h,h=0);
```

$$\sin(a) + a\,\cos(a)$$

```
> limit((f(x)-f(a))/(x-a),x=a);
```

$$\sin(a) + \cos(a)\,a$$

The answers are written differently, but they are obviously the same.

It will be useful to know the derivatives of some of the basic functions that we use. Without comment:

```
> diff(x^8,x);
```

$$8\,x^7$$

```
> diff(6,x);
```

$$0$$

```
> diff(sin(x),x);
```

$$\cos(x)$$

```
> diff(cos(x),x);
```

$$-\sin(x)$$

```
> diff(exp(x),x);
```

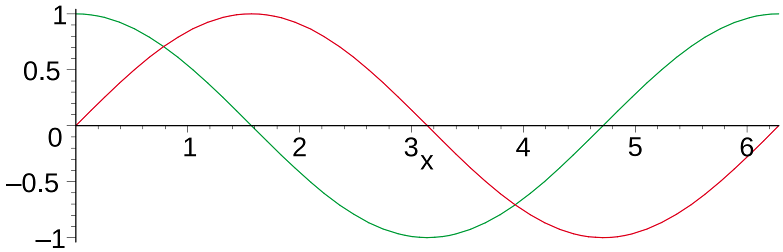
$$\mathbf{e}^x$$

```
> diff(ln(x),x);
```

$$\frac{1}{x}$$

This exercise tells us that the derivative of the sine function is the cosine function. Let's plot both functions on the same coordinate axes. Since we know that $\sin(0) = 0$, we know that the sine graph is below the cosine graph as we start at 0. If we are looking at a color version of these notes, then the sine is the red curve. We put a lot of effort into taking limits that we can now call derivatives. The values that we were seeking were slopes. As we look at the plot which follows, assume that we are on a sled that is going to ride the sine curve. Suppose we had an instrument that would read the numerical values of the cosine (green) curve. Suppose that we had another instrument right next to the first one that would read the slope of the sled at any instant. As we are pulled by a snow-mobile we watch the two instruments and are surprised to see that their readings are the same. If the x and y scales are the same, then our slope at $x = 0$ is 1, at $x = \pi/6$ it will be $\sqrt{3}/2$, at $x = \pi/3$ it will be $1/2$ and at $x = \pi/2$ we should have a slope of 0 because we will be at the crest of our hill. Obviously, we will get negative readings as we head downhill.

```
> plot([sin(x),cos(x)],x=0..2*Pi,color=[red,green],scaling=constrained);
```



Maple Examples: We will use the value of the derivative to determine the slope of our tangent line. We will do two problems, putting the output between them, and side-by-side. Problem 1: $f(x) = \ln(x)$ and $x_0 = \sqrt{2}$.

```
> restart:  with(plots):   with(plottools):
> f:=x->ln(x):
```

$$f := ln$$

```
> x0:=sqrt(2);  y0:=f(x0);
```

$$x0 := \sqrt{2}$$
$$y0 := \frac{1}{2}\ln(2)$$

```
> fprime:=unapply(diff(f(x),x),x);
```

$$fprime := x \to \frac{1}{x}$$

```
> m:=fprime(x0);
```

$$m := \frac{1}{2}\sqrt{2}$$

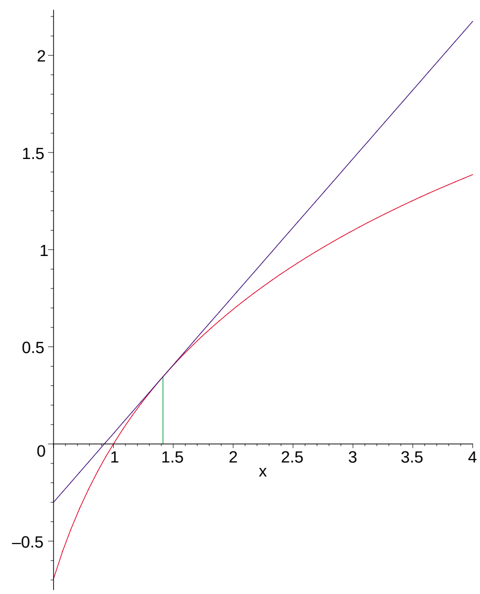
```
> eq1:=y-y0=m*(x-x0);
```

$$eq1 := y - \frac{1}{2}\ln(2) = \frac{1}{2}\sqrt{2}(x - \sqrt{2})$$

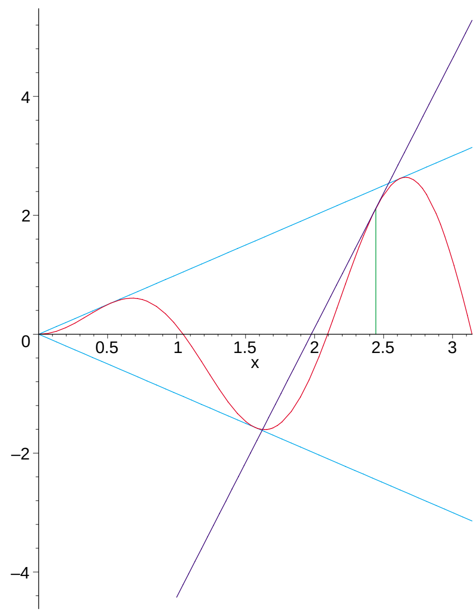
```
> y:=solve(eq1,y);
```

$$y := \frac{1}{2}\ln(2) + \frac{1}{2}\sqrt{2}\,x - 1$$

```
> P1:=plot(f(x),x=1/2..4,color=red):
> P2:=plot(y,x=1/2..4,color=blue):
> P3:=line([x0,0],[x0,f(x0)],color=green):
> display(P1,P2,P3);
```



Problem 1



Problem 2

Problem 2:

```
> restart:   with(plots):   with(plottools):
> g:=x->x*sin(3*x);
```

$$g := c \to x\ \sin(3x)$$

```
> x1:=7*Pi/9;  y1:=g(x1);
```

$$x1 := \frac{7}{9}\pi$$

$$y1 := \frac{7}{18}\pi\sqrt{3}$$

```
> gprime:=unapply(diff(g(x),x),x);
```

$$gprime := x \to \sin(3x) + 3x\cos(3x)$$

```
> m1:=gprime(x1);
```

$$m1 := \frac{1}{2}\sqrt{3} + \frac{7}{6}\pi$$

```
> eq2:=y-y1=m1*(x-x1);
```

$$eq2 := y - \frac{7}{18}\pi\sqrt{3} = (\frac{1}{2}\sqrt{3} + \frac{7}{6}\pi)(x - \frac{7}{9}\pi)$$

```
> y:=solve(eq2,y);
```

$$y := \frac{1}{2}\sqrt{3}\,x + \frac{7}{6}\pi\,x - \frac{49}{54}\pi^2$$

```
> Q1:=plot(y,x=1..Pi,color=blue):
> Q2:=plot(g(x),x=0..Pi,color=red):
> Q3:=plot(x,x=0..Pi,color=cyan):
> Q4:=plot(-x,x=0..Pi,color=cyan):
> Q5:=line([x1,0],[x1,g(x1)],color=green):
> display(Q1,Q2,Q3,Q4,Q5);
```

The output is above the problem. You will note that we included $y = x$ and $y = -x$, which was done because these functions serve as bounding functions for $\sin(3x)$.

As we scan the derivatives again, we see that the derivative of e^x is e^x. What could that possibly mean? If we took a sled ride on the curve $y = e^x$, then the instrument that reads our slope is actually reading our height! Let's try "gussying up" our function by adding a scalar multiple a to x. Now we differentiate that.

```
> diff(exp(a*x),x);
```

$$a\,\mathbf{e}^{(a x)}$$

In **C1M3**, *Exponential Functions*, we had an example involving radioactive decay of a material. We developed an expression that told us the amount present at any time t. Looking back, we see that A was

$$A(t) = 2.837\,\mathbf{e}^{(-.005059468471t)}$$

Using Maple, let's differentiate $A(t)$:

```
> A:=t->(2.837)*exp(-.005059468471*t);
```

$$A := t \to 2.837\,\mathbf{e}^{(-.005059468471t)}$$

```
> diff(A(t),t);
```

$$-.01435371205\,\mathbf{e}^{(-.005059468471t)}$$

```
> -.01435371205/2.837;
```

$$-.005059468470$$

After we take the derivative of $A(t)$, we try something. We divide two coefficients and we get the third. This means that we can write $A'(t)$ as

$$\begin{aligned} A'(t) &= -.01435371205\,\mathbf{e}^{(-.005059468471t)} \\ &= (-.005059468470) \cdot (2.837)\,\mathbf{e}^{(-.005059468471t)} \\ &= (-.005059468470) \cdot A(t) \end{aligned}$$

In other words, the *rate of change* of $A(t)$ is *proportional* to the amount present! The more that you have, the more rapidly it is disappearing.

C1M9 Problems: 1. Use Maple and the **definition** of the derivative (either form) to find $F'(a)$ for $F(x) = \sin(x) + \cos(2x)$.

In problems 2 - 4, use Maple to find the derivatives and to plot the functions with their tangent lines for the given points.

2. $f(x) = \sin(x)\ln(x)$ on $[1/4, 4]$, $x_0 = \pi/3$.

3. $g(x) = x^2 - 3x + 2 + \frac{4}{x}$ on $[1/2, 4]$, $x_1 = 2$.

4. $h(x) = e^x \sin\left(\frac{\pi e^x}{3}\right)$, on $[0, 3]$, $x_2 = 3\ln(2)$.

C1M10

Using the Derivative to Sketch the Function

The behavior of the derivative reveals a lot about the shape of a curve. Everything we know on this topic depends on one theorem which will be discussed later. Because of the importance of this theorem I would like to touch on it now and at least make you aware of what it says geometrically. Some mathematicians refer to it as *The Fundamental Theorem of Differential Calculus* because so much of what we do in beginning calculus depends on it. This theorem is usually known as *The Mean Value Theorem*. We will use a different format than usual by identifying two hypotheses, H1 and H2, and the conclusion C.

The Mean Value Theorem (MVT):

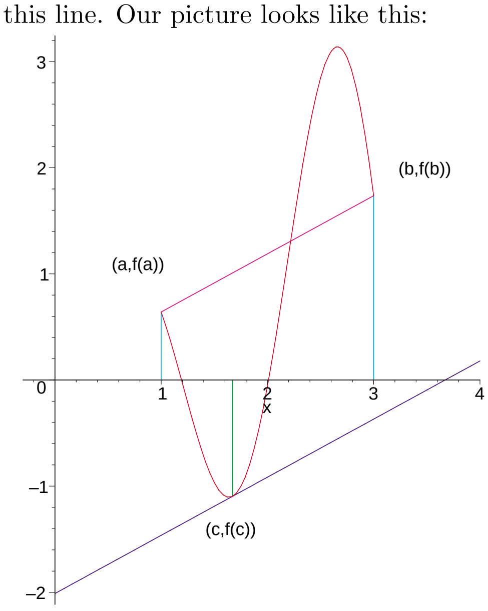
H1: f is a continuous function on the closed interval $[a, b]$.

H2: f is differentiable on the open interval (a, b).

C: There is a point c in (a, b) that satisfies

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Do you remember how we commented earlier that theorems that assert the existence of some mathematical entity are often very strong theorems? First, let's recognize that the number $\dfrac{f(b) - f(a)}{b - a}$ is the slope of the line that joins $(a, f(a))$ and $(b, f(b))$. So, the MVT asserts that there is a tangent line to the curve at some interior point that is parallel to this line. Our picture looks like this:



There are a couple of comments that might help here. There may be several points, $c_1, c_2, \ldots c_n$, that satisfy the condition. The point c MUST NOT be an endpoint.

Maple Example: Use Maple to plot the graph of $f(x) = x\sin(3x) + \dfrac{1}{2}$ on the interval $[1, 3]$ and to draw the line tangent to this curve at the point $(c, f(c))$ where c satisfies the conclusion for the MVT.

Actually, the graph is shown in the diagram above. We will provide the Maple for all but the labeling.

```
> restart:    with(plots):    with(plottools):
> f:=x->x*sin(3*x)+1/2;
```

$$f := x \to x\sin(3x) + \frac{1}{2}$$

```
> a:=1;  b:=3;
```

$$a := 1$$
$$b := 3$$

```
> C:=(f(b)-f(a))/(b-a);
```

$$C := \frac{3}{2}\sin(9) - \frac{1}{2}\sin(3)$$

```
> fprime:=unapply(diff(f(x),x),x);
```

$$fprime := x \to \sin(3x) + 3x\cos(3x)$$

```
> c:=fsolve(fprime(x)=C,x,1..3);
```

$$c := 1.672139733$$

```
> m:=fprime(c);  y0:=f(c);
```

$$m := .5476177153$$
$$y0 := -1.095451597$$

```
> eq1:=y-y0=m*(x-c);
```

$$eq1 := y + 1.095451597 = .5476177153\,x - .9156933402$$

```
> y:=solve(eq1,y);
```

$$y := -2.011144937 + .5476177153\,x$$

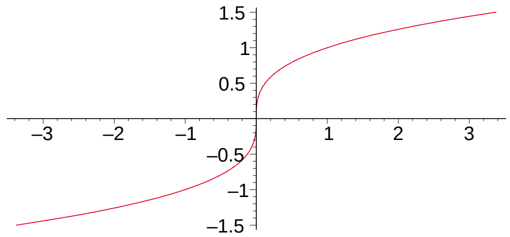
```
> P1:=plot(f(x),x=a..b,color=red,scaling=constrained):
> P2:=plot(y,x=a-1..b+1,color=blue):
> P3:=line([a,f(a)],[b,f(b)],color=magenta):
> P4:=line([c,0],[c,f(c)],color=green):
> P5:=plot(f(x),x=-.3..(-.1),color=white):
> P6:=line([a,0],[a,f(a)],color=cyan):
> P7:=line([b,0],[b,f(b)],color=cyan):
> display(P1,P2,P3,P4,P5,P6,P7);
```

Suppose that $f'(x) > 0$ on (a, b). Then $\frac{f(b)-f(a)}{b-a} > 0$ and $f(b) - f(a) > 0$, or $f(a) < f(b)$. But, using the same argument for $a \le x_1 < x_2 \le b$ and applying the MVT on $[x_1, x_2]$ we see that $f(x_1) < f(x_2)$, or in other words, f is *strictly increasing* on $[a, b]$. Similarly, if $f'(x) < 0$ on $[a, b]$, then f is *strictly decreasing* on $[a, b]$. When the derivative is positive, the function is moving uphill as you move from left to right.

Place your hand so that your fingers are pointing down at about a $45°$ angle in front of you. Assume that there is an imaginary bar a few inches in front of the tip of your fingers. Now move your hand so that it moves under that bar and upwards in a smooth motion. Imagine the slope of your hand to be -1 as you begin, then the slope increases to 0 as you level out, and then it increases to $+1$ as you stop. It is fair to say that the *derivative function* is *increasing*. If we now let the derivative, f', play the role of f in the MVT, then the derivative of the derivative of f must be strictly positive. Let's take a very simple example as our model. Suppose that $f(x) = x^2$. Then $f'(x) = 2x$ and $f''(x) = 2$. The derivative is negative when $x < 0$, 0 at $x = 0$ and positive when $x > 0$. Note that $f''(x) > 0$ for all x. A function that behaves like x^2 is called *concave upwards*.

Now for the opposite situation. If our model is like $g(x) = -x^2$, begin with your hand in front of you pointing upwards at $45°$ and then move your hand smoothly over a bar and downward. The slope was initially about $+1$, it became 0 at the top, and ended at -1. Here the derivative was a *decreasing function*, so the derivative of the derivative must be negative. We say that a function that behaves like $-x^2$ is *concave downwards*.

We will learn later that it is important to discover where a function stops being concave one way and starts being concave the other. Such points occur where the second derivative is zero, or, *does not exist* and are called *inflection points*. Suppose that $p(x) = ax^2 + bx + c$, so that $p'(x) = 2ax + b$ and $p''(x) = 2a$. Then there is no way this quadratic can have an inflection point - the second derivative is a constant, so that p is always concave up $(a > 0)$ or is always concave down $(a < 0)$. In order for a polynomial to have an inflection point the degree must be at least three, so only cubics, quartics, quintics,.., and so forth will have points where the concavity changes. In the figure which follows you see a graph of $y = \sqrt[3]{x} = x^{1/3}$. The derivative does not exist at $x = 0$ (the tangent line would be vertical, which is a "no, no"), but the concavity changes from *up* to *down* as we move from left to right.



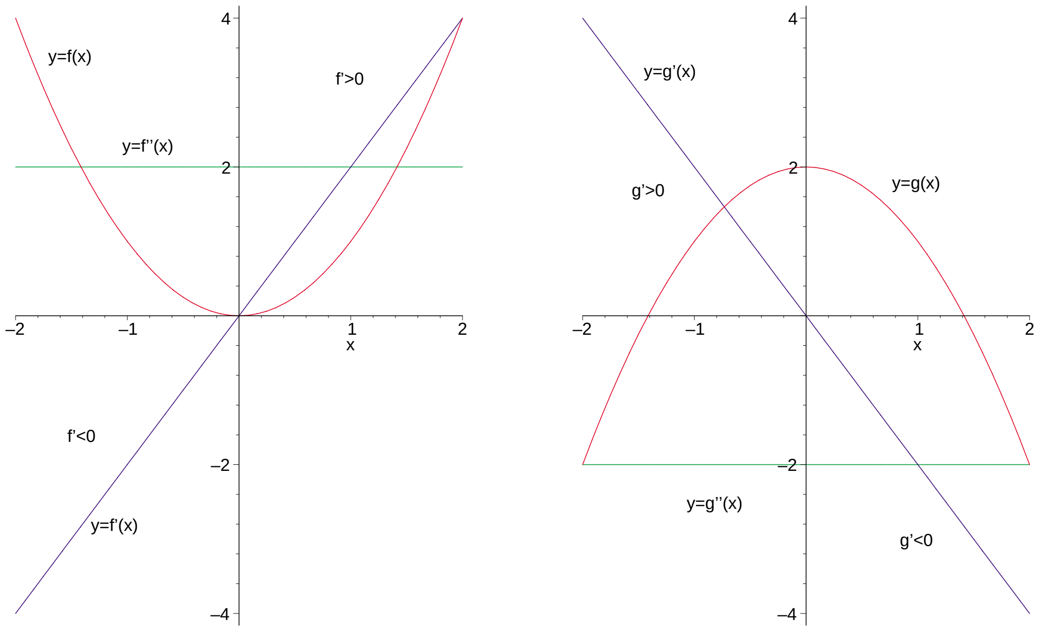
You probably assume that the graph above was produced by

```
> plot(x^(1/3),x=-3.5..3.5);
```

WRONG!! Maple doesn't like taking fractional powers of negative numbers, and rightly so. In order to get this graph we used `rotate`, which is in `plottools`. First we plotted $y = -x^3$, and then we rotated it $90°$.

```
> R:=plot(-x^3,x=-1.5..(1.5),scaling=constrained):
> display(rotate(R,Pi/2));
```

Now let's look at x^2 and $2 - x^2$, and their first and second derivatives.



Concave Upwards **Concave Downwards**

Now we will do another Maple example where we illustrate what happens when the second derivative is positive, zero, and negative. We have drawn vertical dotted lines at the points where the second derivative is zero. Please note how the concavity of F changes at these points. Although we did not make special mention of the behavior of F', its slope is zero at these points, as it should be. And the zeroes of F' correspond to where our function F crests or bottoms out.

Maple Example: For $F(x) = x^4 - 6x^2 + 3$ on $[-3, 3]$, plot F, its derivative, and its second derivative on the same coordinate system and note the relationships.

```
> restart:    with(plots):    with(plottools):
> F:=x->x^4-6*x^2+3;
```

$$F := x \to x^4 - 6x^2 + 3$$

```
> Fprime:=unapply(diff(F(x),x),x);
```

$$Fprime := x \to 4x^3 - 12x$$

```
> solve(Fprime(x)=0,x);
```

$$0, \sqrt{3}, -\sqrt{3}$$

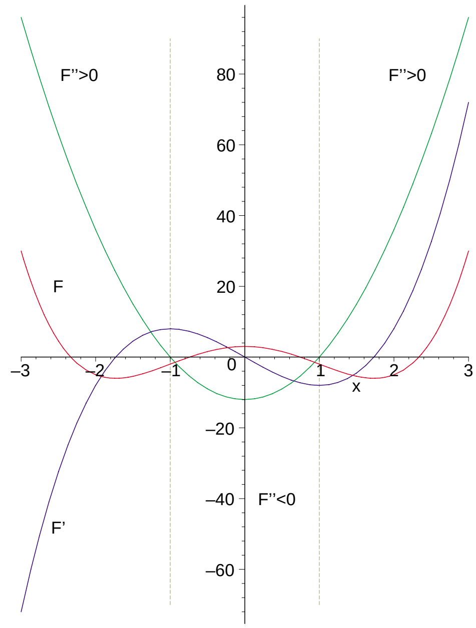
```
> F2prime:=unapply(diff(Fprime(x),x),x);
```

$$F2prime := x \to 12x^2 - 12$$

```
> solve(F2prime(x)=0,x);
```

$$1, -1$$

```
> S1:=plot(F(x),x=-3..3,color=red):
> S2:=plot(Fprime(x),x=-3..3,color=blue):
> S3:=plot(F2prime(x),x=-3..3,color=green):
> S4:=textplot([-2.5,20.2,"F"]):
> S5:=textplot([-2.5,-48,"F'"]):
> S6:=textplot([-2.2,80,"F''>0"]):
> S61:=textplot([.45,-40,"F''<0"]):
> S62:=textplot([2.2,80,"F''>0"]):
> S7:=line([-1,-70],[-1,90],color=khaki,linestyle=3):
> S8:=line([1,-70],[1,90],color=khaki,linestyle=3):
> display(S1,S2,S3,S4,S5,S6,S61,S62,S7,S8);
```



C1M10 Problems:

1. Given $f(x) = x^3 - x + 2$ on $[-1, 2]$. Using the first Maple Example as a prototype, plot the graph of $f(x)$ and the tangent line at the point which satisfies the conclusion of the Mean Value Theorem.

2. Given $g(x) = 3x^4 - 4x^3$ on $[-5/4, 2]$. Use Maple to plot $g(x)$, $g'(x)$, and $g''(x)$.

3. Given $h(x) = \cos(x) + \sin(2x)$ on $[-2, 2]$. Use Maple to plot $h(x)$, $h'(x)$, and $h''(x)$. Use `fsolve` to find those values where $h''(x)$ is 0 and put vertical lines there. Remember, you can specify the range for the solution in `fsolve`, as we did in one of the examples.

C1M11

Differentiation: Rules and Review

This begins with a modification of part of the review we do for those students who begin with Calculus II their first semester. We remind you of the definition and then list the rules. The notation we use for the derivative of $f(x)$ with respect to x is $D_x\big(f(x)\big) = f'(x)$.

Definition of derivative. Suppose f is defined on an open interval containing x. The derivative of f at x is defined by

$$D_x\left(f(x)\right) = f'(x) \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Tangent line. If $f(x_0) = y_0$ and $f'(x_0) = m$, then an equation for the line tangent to the curve $y = f(x)$ is given by

$$y - y_0 = m(x - x_0)$$

Rules of Differentiation Assume that a and b are real numbers and that $f(x)$ and $g(x)$ are differentiable on an open interval containing x.

Rule 1. The derivative is linear. That is, $D_x\big(af(x) + bg(x)\big) = aD_x(f(x)) + bD_x(g(x)) = af'(x) + bg'(x)$.

Rule 2. Product Rule. $D_x\big(f(x)g(x)\big) = D_x(f(x))g(x) + f(x)D_x(g(x)) = f'(x)g(x) + f(x)g'(x)$.

Rule 3. Quotient Rule. On an interval where $g(x) \neq 0$,

$$D_x\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_x\big(f(x)\big) - f(x)D_x\big(g(x)\big)}{\big(g(x)\big)^2} = \frac{g(x)f'(x) - f(x)g'(x)}{\big(g(x)\big)^2}$$

Rule 4. Power function derivative. If r is a real number, then

$$D_x(x^r) = rx^{r-1}$$

Examples: $D_x\left(x^{4/3}\right) = \dfrac{4}{3}x^{1/3}$, $\quad D_x\left(\dfrac{1}{x^{4/3}}\right) = -\dfrac{4}{3}\dfrac{1}{x^{7/3}}$, $\quad D_x\big(\sqrt{x}\,\big) = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

Rule 5. Chain Rule. On an open interval for which $(f \circ g)(x) \equiv f\big(g(x)\big)$ is defined

$$D_x\big((f \circ g)(x)\big) = D_x\Big(f\big(g(x)\big)\Big) = f'(g(x)) \cdot g'(x)$$

Example: $D_x\Big(\big(4 + x^3\big)^5\Big) = (5)\big(4 + x^3\big)^4(3x^2)$

Rule 6. Reciprocal Rule. On an interval where $f(x) \neq 0$ we have

$$D_x\left(\frac{1}{f(x)}\right) = \frac{-f'(x)}{\big(f(x)\big)^2}$$

Derivatives of trigonometric functions.

Using Maple earlier we showed that $D_x\big(\sin(x)\big) = \cos(x)$ and $D_x\big(\cos(x)\big) = -\sin(x)$. Let's use these facts and the rules to find the other trig derivatives.

1. Using the Reciprocal Rule,

$$D_x\big(\sec(x)\big) = D_x\left(\frac{1}{\cos(x)}\right) = \frac{-D_x\big(\cos(x)\big)}{\big(\cos(x)\big)^2} = \frac{-\big(-\sin(x)\big)}{\cos^2(x)} = \left(\frac{1}{\cos(x)}\right)\left(\frac{\sin(x)}{\cos(x)}\right) = \sec(x)\tan(x)$$

2. Using the Quotient Rule,

$$\begin{aligned} D_x\big(\tan(x)\big) = D_x\left(\frac{\sin(x)}{\cos(x)}\right) &= \frac{\cos(x)\cdot D_x\big(\sin(x)\big) \;-\; \sin(x)\cdot D_x\big(\cos(x)\big)}{\big(\cos(x)\big)^2} \\ &= \frac{\cos(x)\cos(x) \;-\; \sin(x)\big(-\sin(x)\big)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x) \end{aligned}$$

In a similar manner we obtain

$$D_x\big(\csc(x)\big) = -\csc(x)\cot(x) \qquad\qquad D_x\big(\cot(x)\big) = -\csc^2(x)$$

Let's develop a means of remembering this information in a systematic manner. We are going to form a four-column table by listing the basic functions in the first column, their derivatives in the second column, and their cofunctions in the third column. The fourth column will eventually contain the derivatives of the cofunctions, but for now we leave it blank.

Function	Derivative	Cofunction	Derivative
$\sin(x)$	$\cos(x)$	$\cos(x)$	
$\tan(x)$	$\sec^2(x)$	$\cot(x)$	
$\sec(x)$	$\sec(x)\tan(x)$	$\csc(x)$	

Then, put a minus sign in each of the remaining boxes. Remember, the derivative of a **CO**-function always gets a **minus** sign!

Function	Derivative	Cofunction	Derivative
$\sin(x)$	$\cos(x)$	$\cos(x)$	$-$
$\tan(x)$	$\sec^2(x)$	$\cot(x)$	$-$
$\sec(x)$	$\sec(x)\tan(x)$	$\csc(x)$	$-$

Complete the table by putting the cofunctions of the entries in the second column into the fourth column.

Function	Derivative	Cofunction	Derivative
$\sin(x)$	$\cos(x)$	$\cos x$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$	$\cot(x)$	$-\csc^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$	$\csc(x)$	$-\csc(x)\cot(x)$

This is an excellent memory device. However, you **must** know the derivatives of the basic trig functions $\sin(x)$, $\tan(x)$, and $\sec(x)$, in order to complete the table.

As a very simple consequence of the Chain Rule, it is useful to list:

$$D_x\big(\sin(ax)\big) = a\,\cos(ax) \qquad\qquad D_x\big(\cos(ax)\big) = -a\,\sin(ax)$$

♠ It is important to remember when applying the Chain Rule to trigonometric, and other functions:
THE ARGUMENT NEVER CHANGES!

♠ When differentiating a composite function with respect to x, the last step in the Chain Rule is to differentiate some element *with respect to* x.

Example: Using the Chain Rule,

$$\begin{aligned} D_x\big(\tan^3(\sqrt{x}\,)\big) &= 3\tan^2(\sqrt{x}\,)\cdot D_x\big(\tan(\sqrt{x}\,)\\ &= 3\tan^2(\sqrt{x}\,)\cdot \sec^2(\sqrt{x}\,)\cdot D_x(\sqrt{x}\,)\\ &= 3\tan^2(\sqrt{x}\,)\cdot \sec^2(\sqrt{x}\,)\cdot\left(\frac{1}{2\sqrt{x}}\right)\end{aligned}$$

Basically, this is u^3 with $u=\tan(\sqrt{x}\,)$, resulting in $3u^2\cdot D_x(u)$. The argument of the tangent function is $\sqrt{x}$, so when the tangent is differentiated, the result is the secant squared *of the same argument.* This produced a factor of $\sec^2(\sqrt{x}\,)$. Finally, we differentiate $\sqrt{x}$ *with respect to* x.

Derivative of exponential and logarithmic functions. You may recall that earlier we used Maple to see what the derivatives of these functions would be. We found

$$D_x\big(e^x\big) = e^x \qquad D_x(\ln x) = \frac{1}{x},\ x>0 \qquad D_x\big(a^x\big) = (\ln a)a^x,\ a>0$$

If $x<0$, then we will use the Chain Rule to look at

$$D_x\big(\ln(-x)\big) = \left(\frac{1}{-x}\right)D_x(-x) = \left(\frac{1}{-x}\right)(-1) = \frac{1}{x}$$

This allows us to include the derivative of $\ln|x|$.

$$D_x(\ln|x|) = \frac{1}{x},\ x\neq 0$$

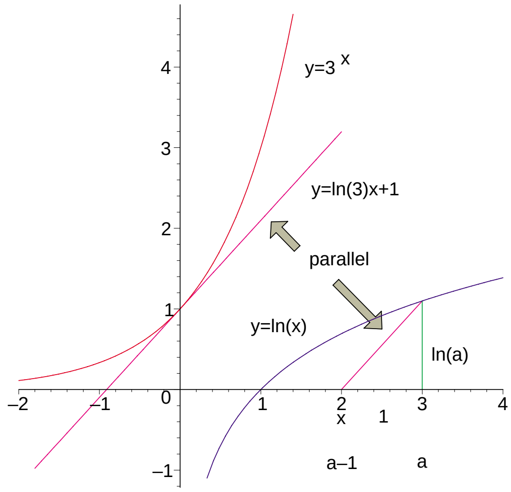
There is an identity that I have found to be extremely useful when dealing with exponential functions. Namely,

$$a^b = e^{b\,\ln(a)}$$

Using this, start with a^x. Then, $a^x = e^{\ln(a)\,x}$, and taking the derivative (we know $D_x(e^{cx}) = c\,e^{cx}$ by the Chain Rule) we get

$$D_x\big(a^x\big) = D_x\big(e^{\ln(a)\,x}\big) = \ln(a)\,e^{\ln(a)\,x} = \ln(a)\,a^x$$

Because $a^0 = 1$ for all positive a, every function of the form $y = a^x$ passes through $(0,1)$ and the slope of the tangent line there is always $\ln(a)$. If we draw a line from $(a-1,0)$ to $(a,\ln(a))$, then this line must be parallel to the tangent line we just discussed. The figure which follows illustrates this concept for $a = 3$.



We will limit our discussion on inverse trigonometric functions to

$$\sin^{-1} x \equiv \arcsin x \qquad \text{and} \qquad \tan^{-1} x \equiv \arctan x$$

We remind the reader that the exponents refer to inverse functions and **not** to reciprocals. The derivatives are listed below:

$$D_x(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad\qquad D_x(\arctan x) = \frac{1}{1+x^2}$$

Examples: (a) Find $D_x\big(\arcsin(3x) - \sqrt{1-9x^2}\big)$.

This is a straightforward application of linearity and the Chain Rule.

$$D_x\big(\arcsin(3x)\big) = \frac{1}{\sqrt{1-(3x)^2}} \cdot D_x(3x) = \frac{1}{\sqrt{1-9x^2}} \cdot 3$$

$$D_x\big(\sqrt{1-9x^2}\big) = \left(\frac{1}{2}\right) \cdot \big(1-9x^2\big)^{-1/2} \cdot D_x(1-9x^2) = \left(\frac{1}{2}\right) \cdot \frac{-18x}{\sqrt{1-9x^2}} = \frac{-9x}{\sqrt{1-9x^2}}$$

Combining both parts, we have

$$D_x\big(\arcsin(3x) - \sqrt{1-9x^2}\big) = \frac{3}{\sqrt{1-9x^2}} - \frac{-9x}{\sqrt{1-9x^2}} = \frac{3+9x}{\sqrt{1-9x^2}}$$

(b) Find $D_x\left(3\arctan\left(\dfrac{x}{3}\right) + 4\ln(x^2+9)\right)$

Taking the derivative we get

$$3 \cdot \frac{1}{1+\left(\frac{x}{3}\right)^2} \cdot \left(\frac{1}{3}\right) + 4 \cdot \left(\frac{1}{x^2+9}\right) \cdot (2x) = \frac{9}{9+9\cdot\frac{x^2}{9}} + \frac{4\cdot 2x}{x^2+9} = \frac{9}{9+x^2} + \frac{8x}{x^2+9} = \frac{8x+9}{x^2+9}$$

C1M11 Problems:

1. Find the derivative with respect to x of the following and check your answers using Maple:

a. $\dfrac{1}{\ln(x)}$ b. $\ln\big(\cos(x)\big)$ c. $\dfrac{1}{\sqrt{1-x^2}}\arcsin(x)$

2. Find the derivative with respect to x of the following and check your answers using Maple:

a. $e^{\cos(2x)}$ b. $\sin\left(e^{3x}\right)$ c. $\dfrac{\sin(2x)}{\cos(3x)}$

3. Use Maple to plot $y = \arcsin(x)$ and to evaluate $\lim\limits_{x\to -1^+} \arcsin(x)$ and $\lim\limits_{x\to 1^-} \arcsin(x)$

4. Use Maple to plot $y = \arctan(x)$ and to evaluate $\lim_{x\to\infty} \arctan(x)$ and $\lim_{x\to-\infty} \arctan(x)$

C1M12

Extrema

A while back in **C1M10** we discussed the Mean Value Theorem, but only lightly. We used it to determine that a continuous function must be increasing on an interval where the derivative is positive,. Similarly, a negative derivative signals a decreasing function. Let's recall the two hypotheses of the MVT and then add a third.

H1: f is a continuous function on the closed interval $[a, b]$.

H2: f is differentiable on the open interval (a, b).

H3: There is a point c in (a, b) for which $f(x) \leq f(c)$ for all x in some open interval containing c.

The conclusion will be stated below. We are going to approach c from each side and compare the results.

Assume $x < c$. Then $\dfrac{f(x) - f(c)}{x - c} \geq 0$, so $\lim_{x\to c^-} \dfrac{f(x) - f(c)}{x - c} = f'(c) \geq 0$.

Assume $x > c$. Then $\dfrac{f(x) - f(c)}{x - c} \leq 0$, so $\lim_{x\to c^+} \dfrac{f(x) - f(c)}{x - c} = f'(c) \leq 0$.

Taken together, $f'(c) \geq 0$ and $f'(c) \leq 0$ imply $f'(c) = 0$. The one-sided limits existed because $f'(c)$ existed. When there is an open interval containing a point and that point satisfies $f(x) \leq f(c)$ for all x in the interval, we say that f has a *local maximum* or *relative maximum* at c.

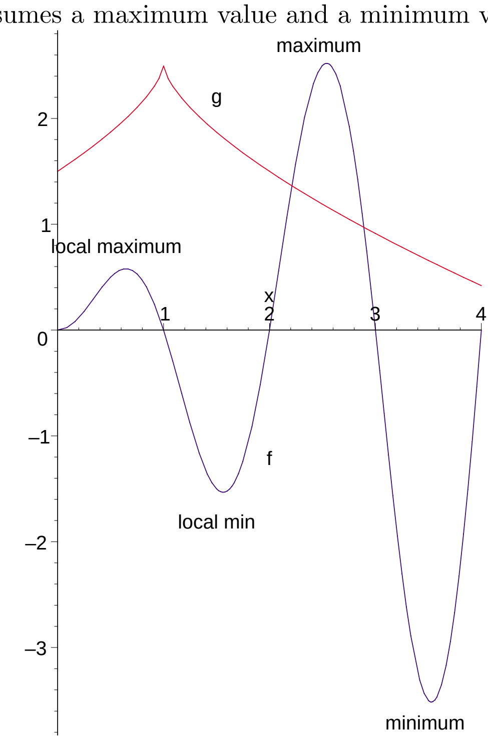
In a similar manner, suppose that we replace **H3** with

H4: There is a point c in (a, b) for which $f(x) \geq f(c)$ for all x in some open interval containing c.

In this case f has a *local minimum* or *relative minimum* at c. By an obvious parallel argument we have the same conclusion. So, with either **H3** or **H4** our conclusion is :

C: $f'(c) = 0$.

This tells us that when our function is differentiable the only way we can have a *local extreme* (local maximum or local minimum) that is not an endpoint, is for the derivative to be zero at the point. For this reason we define a point c to be a *critical point* if either the derivative does not exist at c or $f'(c) = 0$. The Extreme Value Theorem was mentioned in the module on continuity. From it we learned that a continuous function on a closed interval assumes a maximum value and a minimum value.



Here we have the graphs of two continuous functions on $[0,4]$, f and g. We have labeled the extrema of f, except for the endpoints which are a local minimum ($x=0$) and a local maximum ($x=4$). It is clear that at $x=1$ the function g has a maximum, but the derivative does not exist there. The endpoints serve as a local minimum ($x=0$) and a minimum (absolute) ($x=4$) for g.

Maple Example: The function f in the diagram above is actually defined by $f(x) = x\sin(\pi x)$. Let's use Maple to locate all extrema of f. We will show all steps, including the plotting of text. After finding our first relative maxima, at c_1, we will test the values of f on both sides of c_1. We will also test the derivative of f on both sides.

```
> restart:    with(plots):  with(plottools):
> f:=x->x*sin(Pi*x);  a:=0;  b:=4;
```

$$f := x \to x\,\sin(\pi x)$$
$$a := 0$$
$$b := 4$$

```
> fprime:=unapply(diff(f(x),x),x);
```

$$fprime := x \to \sin(\pi x) + x\cos(\pi x)\pi$$

This command takes the derivative of $f(x)$ with respect to x and sets it up as a function.

```
> f2prime:=unapply(diff(f(x),x,x),x);
```

$$f2prime := x \to 2\,\cos(\pi x)\pi - x\,\sin(\pi x)\pi^2$$

Note how this command takes the *second* derivative of $f(x)$ and sets it up as a function.

```
> c1:=fsolve(fprime(x)=0,x=0..1);
```

$$c1 := .6457736765$$

```
> m1:=evalf(f(c1));
```

$$m1 := .5792303274$$

```
> evalf(f(c1-.02));  evalf(f(c1+.02));
```

$$.5775557266$$
$$.5775085314$$

```
> evalf(fprime(c1-.02));  evalf(fprime(c1+.02));
```

$$.1662069815$$
$$-.1732844003$$

From the last few lines we see that the derivative of f is positive on the left of c_1 and negative on the right of c_1. This means that as we move from left to right we are going uphill, level, and then downhill $-$ a sure sign that c_1 yields a relative maximum. Later we will state all this as a theorem.

```
> evalf(f2prime(c1));
```

$$-8.494699804$$

This is overkill, but we may also use the second derivative to see if the first derivative is a decreasing function, and the negative value of `f2prime` at c_1 says that it is, thus signalling that f is concave down at c_1 and there is a relative maximum there.

```
> c2:=fsolve(fprime(x)=0,x=1..2);
```

$$c2 := 1.563913906$$

```
> m2:=evalf(f(c2));
```

$$m2 := -1.532493362$$

```
> evalf(f2prime(c2));
```

$$16.37825282$$ Concave up, relative minimum

```
> c3:=fsolve(fprime(x)=0,x=2..3);
```

$$c3 := 2.539688175$$

```
> m3:=evalf(f(c3));
```

$$m3 := 2.519972588$$

```
> evalf(f2prime(c3));
```

$$-25.65251748$$ Concave down, relative maximum

```
> c4:=fsolve(fprime(x)=0,x=3..4);
```

$$c4 := 3.528636468$$

```
> m4:=evalf(f(c4));
```

$$m4 := -3.514366510$$

```
> evalf(f2prime(c4));
```

$$35.24990626$$ Concave up, relative minimum

```
> A:=f(a); B:=f(b);
```

$$A := 0$$
$$B := 0$$

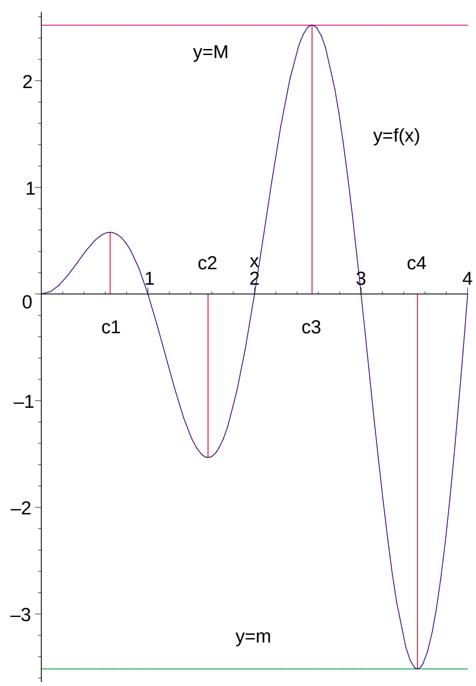
```
> M:=max(m1,m2,m3,m4,A,B);
```

$$M := 2.519972588$$

```
> m:=min(m1,m2,m3,m4,A,B);
```

$$m := -3.514366510$$

```
> B1:=plot(f(x),x=0..4,color=blue,scaling=constrained):
> B2:=line([c1,0],[c1,f(c1)],color=red):
> B3:=line([c2,0],[c2,f(c2)],color=red):
> B4:=line([c3,0],[c3,f(c3)],color=red):
> B5:=line([c4,0],[c4,f(c4)],color=red):
> B21:=textplot([c1,-.3,"c1"]):
> B31:=textplot([c2,.3,"c2"]):
> B41:=textplot([c3,-.3,"c3"]):
> B51:=textplot([c4,.3,"c4"]):
> B6:=plot(m,x=0..4,color=green):
> B61:=textplot([2,m+.3,"y=m"]):
> B7:=plot(M,x=0..4,color=magenta):
> B71:=textplot([1.6,M-.24,"y=M"]):
> B8:=textplot([3.35,1.5,"y=f(x)"]):
> display(B1,B2,B3,B4,B5,B21,B31,B41,B51,B6,B61,B7,B71,B8);
```



Maple Example: Find the minimum distance from $Q(3/2,1)$ to the graph of $f(x) = x\,\sin(\pi x)$.

Frequently calculus texts include an exercise to find the minimum distance from a given point to a simple curve, such as a parabola. This is a similar, yet more challenging, problem involving the function in the previous example. We will exploit the power of Maple and have it find numerical solutions for critical points that we would be unlikely to locate easily. The function which determines the distance from the point $Q(3/2,1)$ to the point $(x, f(x))$ on the curve is designated as $R(x)$, its derivative is $S(x)$, and its second derivative is $T(x)$. Because $T(x)$ is a bit cumbersome, its display is suppressed with a colon. Since several points may occur as solutions, we will use the Second Derivative Test to determine whether a critical point is a relative maximum or minimum. The derivative has a denominator that looks as if it could be a problem. However, a close examination reveals that it is simply the distance function $R(x)$. Since Q does not lie on the curve, the denominator won't be 0, but we will check it anyway. We chose to make the derivatives of f functions instead of expressions.

```
> restart:    with(plots):    with(plottools):
> f:=x->x*sin(Pi*x);
```

$$f := x \to x\,\sin(\pi x)$$

```
> Q:=[3/2,1];
```

$$Q := \left[\frac{3}{2}, 1\right]$$

Set up our distance formula.

```
> R:=x->sqrt((x-Q[1])^2+(f(x)-Q[2])^2);
```

$$R := x \to \sqrt{(x - Q_1)^2 + (f(x) - Q_2)^2}$$

Set up the first derivative as a function.

```
> S:=unapply(diff(R(x),x),x);
```

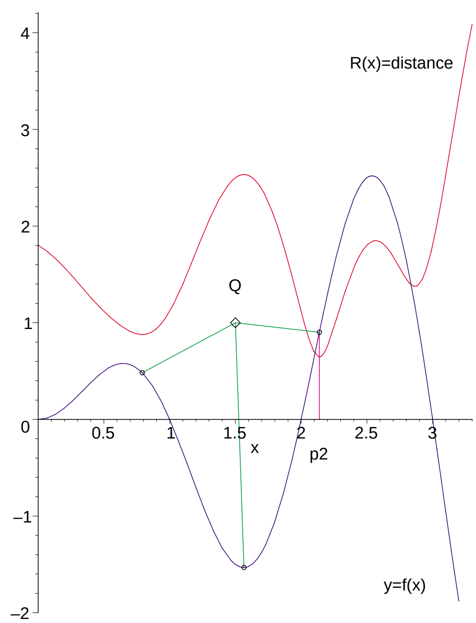
$$S := x \to \frac{1}{2}\,\frac{(2x - 3 + 2(x\,\sin(\pi x) - 1)(\sin(\pi x) + x\,\cos(\pi x)\,\pi))}{\sqrt{(x - 3/2)^2 + (x\,\sin(\pi x) - 1)^2)}}$$

Let's take a look at that denominator. If it has a zero in the interval, then we have a problem. We will "eyeball" it rather than solving the equation.

```
> De:=denom(S(x));
```

$$2(\sqrt{(x - 3/2)^2 + (x\,\sin(\pi x) - 1)^2})$$

```
> D1:=plot(De,x=0..3.2,color=red):
> D2:=plot(S(x),x=0..3.2,color=blue):
> D3:=textplot([1.55,6,"denominator of derivative"]):
> D4:=textplot([1.5,-3.5,"derivative"]):
> display(D1,D2,D3,D4);
```



Set up the second derivative of the function as a function.

```
> T:=unapply(diff(R(x),x,x),x):
> P0:=plot(f(x),x=0..3.2,color=blue):
> p1:=fsolve(S(x)=0,x,1/2..5/2);
```

$$p1 := 1.565491407$$

```
> evalf(T(p1));
```

$$-16.00806205$$

$T(p1) = R''(p1) < 0 \;\Rightarrow$ relative maximum

```
> P1:=pointplot([p1,f(p1)],symbol=circle):
> p2:=fsolve(S(x)=0,x,1.6..5/2);
```

$$p2 := 2.138585926$$

```
> evalf(T(p2));
```

$$67.70258030$$

$T(p2) = R''(p2) > 0 \;\Rightarrow$ relative minimum

```
> P2:=pointplot([p2,f(p2)],symbol=circle):
> p3:=fsolve(S(x)=0,x,.5..1.4);
```

$$p3 := .7918557211$$

```
> evalf(T(p3));
```

$$9.019878695$$

$T(p3) = R''(p3) > 0 \;\Rightarrow$ relative minimum

```
> P3:=pointplot([p3,f(p3)],symbol=circle):
> A:={evalf(R(p1)),evalf(R(p2)),evalf(R(p3))};
```

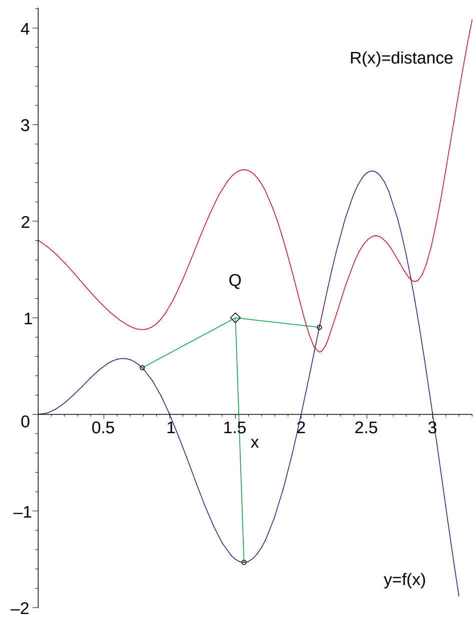
$$A := \{2.533319655, .6460679294, .8775682312\}$$

```
> mindistance:=min(op(A));
```

$$\mathit{mindistance} := .6460679294$$

We used `op(A)` in order to access the components of the set A. We have our answer and the point at which it occurs, $(p2, f(p2))$. Now we will display a graph that illustrates the items which we found.

```
> P4:=pointplot(Q,symbol=diamond,symbolsize=15):
> P5:=line(Q,[p1,f(p1)],color=green):
> P6:=line(Q,[p2,f(p2)],color=green):
> P7:=line([p2,f(p2)],[p2,0],color=magenta):
> P8:=line(Q,[p3,f(p3)],color=green):
> P9:=plot(R(x),x=0..3.3,color=red):
> T1:=textplot([2.8,-1.7,"y=f(x)"]):
> T2:=textplot([2.8,3.7,"R(x)=distance"]):
> T3:=textplot([1.5,1.4,"Q"]):
> T4:=textplot([p2,-.35,"p2"]):
> display(P0,P1,P2,P3,P4,P5,P6,P7,P8,P9,T1,T2,T3,T4);
```



Procedure to Locate Extrema: It is time to summarize our approach to extrema. Suppose that we have a continuous function f on a closed interval. We locate all points at which the first derivative of f is zero or for which the derivative does not exist, calling such points *critical points*. We evaluate f at the critical points and the endpoints and know that our global maximum and global minimum will be found among these values. To determine whether a critical point for which f is differentiable yields a relative maxima or minima, we apply the First Derivative Test.

First Derivative Test: Assume that f is differentiable on an open interval containing c and that $f'(c) = 0$.

(a) If the derivative $f'(x)$ changes sign from positive to negative as x moves through c from left to right, then f has a relative maximum at c.

(a) If the derivative $f'(x)$ changes sign from negative to positive as x moves through c from left to right, then f has a relative minimum at c.

Noting that in case (a) the derivative is decreasing and that in case (b) the derivative is increasing, we state the Second Derivative Test for determining the status of critical points where the derivative is zero.

Second Derivative Test: Assume that f is differentiable on an open interval containing c and that $f'(c) = 0$. Assume that the second derivative, f'', also exists on the open interval.

(a) If $f''(c) < 0$, then f has a relative maximum at c.

(b) If $f''(c) > 0$, then f has a relative minimum at c.

There is another concept to consider that we have not yet mentioned. What happens when $f''(c) = 0$? The answer is inconclusive without additional information. A *point of inflection* occurs when a function stops being concave one way and starts being concave the other way. This happens when the derivative stops increasing and starts decreasing, or vice versa.

Point of Inflection: Suppose that $f''(x)$ exists on an open interval containing c and that $f''(c) = 0$. Then c is a point of inflection of f if $f''(x)$ changes sign as x passes through c.

Examples: (a) If $f(x) = x^4$, then $f'(x) = 4x^3$ and $f''(x) = 12x^2$. If $c = 0$ then $f''(c) = 0$, but c is **not** a point of inflection for f because f'' does not change sign as x passes through 0.

(b) Suppose that $f''(x) = (x-1)^2(x-2)$. Then f'' is zero at 1 and 2. But f'' changes sign only at 2, so 2 produces the only inflection point for f.

Maple Example: Find all points of inflection of $f(x) = 3x^5 - 10x^4 + 10x^3 - 60x^2 + 5x + 2$ and plot the graph of f.

```
> restart:    with(plots):    with(plottools):
> f:=x->3*x^5-10*x^4+10*x^3-60*x^2+5*x+2;
```

$$f := x \to 3x^5 - 10x^4 + 10x^3 - 60x^2 + 5x + 2$$

```
> f2prime:=unapply(diff(f(x),x,x),x);
```

$$f2prime := x \to 60\,x^3 - 120\,x^2 + 60\,x - 120$$

```
> factor(f2prime(x));
```

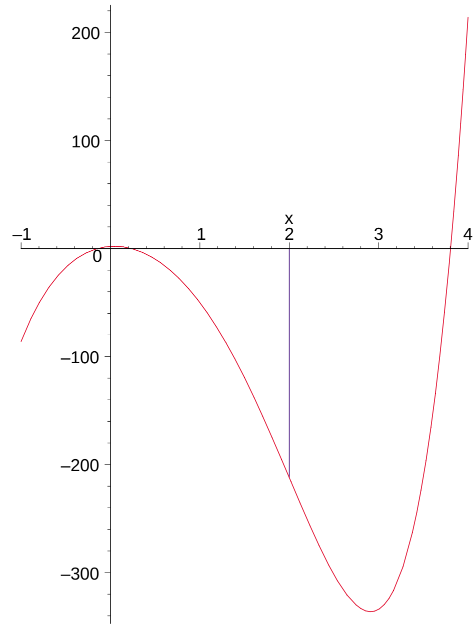
$$60\,(x-2)\,(x^2+1)$$

```
> f2prime(1.98);  f2prime(2.02);
```

$$-5.904480$$
$$6.096480$$

At 2, f stops being concave down and starts being concave up.

```
> U1:=plot(f(x),x=-1..4):
> U2:=line([2,0],[2,f(2)],color=blue):
> display(U1,U2);
```



C1M12 Problems:

1. Use Maple to find the critical points, points of inflection, and to display the graphs of the given functions, their derivatives and second derivatives.

a. $f(x) = x^4 + 6x^3 - 24x^2 + 24$ b. $g(x) = 3x^5 - 5x^3$

2. Given: $f(x) = x\ \cos(\pi x)$ on $[1, 5]$.

a. Use Maple to find the relative maxima and minima and the global maximum and minimum as was done in the Maple Example. Display vertical lines at the extrema as in the example.

b. Use Maple to locate the point $P(x, y)$ on the graph of $f(x)$ that is closest to the point $Q(3, 5)$. Display lines connecting the possible solutions to Q as was done in the Maple Example.

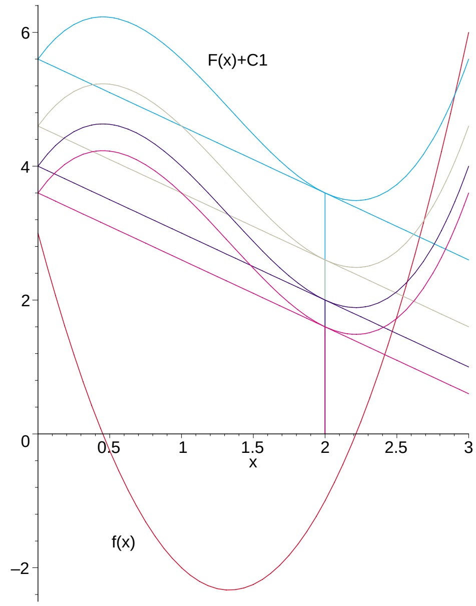
C1M13

Antiderivatives

Fortunately, the topic of *antiderivatives* has nothing to do with social unrest and rebellion within the ranks of mathematicians. It refers to reversing the process of differentiation and determining a new function whose derivative is the function at hand. The social part comes in because when you find one antiderivative you actually get an entire family of functions that serve as antiderivatives of the function. Think about it. If $F'(x) = f(x)$, then $F(x)$ is *an* antiderivative of $f(x)$, and for any constant C, $F(x) + C$ has $f(x)$ as its derivative with respect to x also. Let's formalize this a little.

Definition: A function F is called an *antiderivative* of a function f on an interval I if $F'(x) = f(x)$ for all x in I.

In the plot below you will find a parabola labeled $f(x)$ and a family of cubics with one labeled $F(x)+C1$. Each cubic has $f(x)$ as its derivative, so they are all *antiderivatives of* $f(x)$. We have shown the tangent at $x = 2$ for each and note that the parallel lines all have a slope of -1.



There is a theorem that is usually mentioned in a section on differentiation. Basically, it states that if two functions are defined on an open interval and have the same derivative at all points of that interval, then the two functions differ by a constant. Let's set this up as hypotheses and conclusion.

Theorem:

H1: Functions f and g are defined on an open interval I.

H2: $f'(x) = g'(x)$ for all x in I.

C: There is a constant C for which $f(x) = g(x) + C$ for all x in I.

What does this mean to us here? It means that if we have a collection of functions, all of which are antiderivatives of the same function on an open interval, then each member differs from another by a constant. They are all "shaped the same" and you can get one from the other by translating it up or down. This is true because $f'(x) = g'(x) \Rightarrow (f' - g')(x) = f'(x) - g'(x) = 0$ and only constant functions have zero for their derivative. This last statement is yet another consequence of the **MVT**.

Maple Example: Use Maple to find an antiderivative $F(x)$ of $f(x) = e^{2x}$ that satisfies $F(1.2) = 4$. The condition $F(1.2) = 4$ is called a *boundary value* for F.

The terms *antiderivative* and *integral* are commonly used interchangeably even though that usage is imprecise. Maple uses `int` as a command to find either for a function. The use of a lowercase "i" makes the command active, while an uppercase "I" is inactive and some form of evaluation is necessary to activate it.

```
> restart:    with(plots):    with(plottools):
> f:=x->exp(2*x);
```

$$f := x \to \mathbf{e}^{(2x)}$$

Now we will find an antiderivative for $f(x)$.

```
> A:=int(f(x),x);
```

$$A := \frac{1}{2}\,\mathbf{e}^{(2x)}$$

```
> F:=unapply(A+C,x);
```

$$F := x \to \frac{1}{2}\,\mathbf{e}^{(2x)} + C$$

This is how we add a constant to our expression and then make a function out of the result. Next we force $F(1.2) = 4$ and solve for the constant C that satisfies this condition.

```
> C:=solve(F(1.2)=4,C);
```

$$C := -1.511588190$$

```
> F(x);
```

$$\frac{1}{2}\,\mathbf{e}^{(2x)} - 1.511588190$$

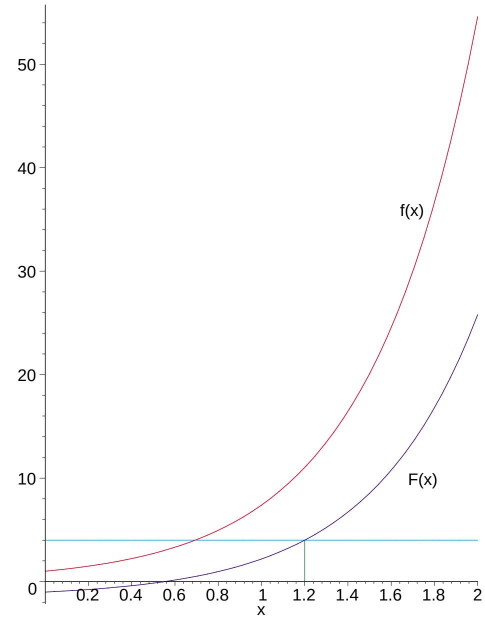
```
> evalf(F(1.2));
```

$$4.000000000$$

```
> diff(F(x),x);
```

$$\mathbf{e}^{(2x)}$$

The last two lines are simply to check that we got the results that we wanted. You should be able to produce the plot which follows.



Maple Example: A stone is thrown upwards from a cliff with a velocity of 48 feet per second and it lands 1120 feet below. Using the point where it lands as the reference point, determine when the stone lands and how fast it is going. Gravitational acceleration is 32 feet per second per second, *downwards*. This means that we are given *two* boundary values. If $a(t)$ denotes acceleration and $v(t)$ denotes velocity, then

$$a(0) = -32 \qquad\qquad v(0) = 48$$

```
> restart:
> a:=-32;
```

$$a := -32$$

```
> v:=unapply(int(a,t)+C1,t);
```

$$v := t \to -32\,t + C1$$

Antidifferentiate $a(t)$, add a constant, make v into a function of t.

```
> C1:=solve(v(0)=48,C1);
```

$$C1 := 48$$

```
> v(t);
```

$$-32\,t + 48$$

```
> p:=unapply(int(v(t),t)+C2,t);
```

$$p := t \to -16\,t^2 + 48\,t + C2$$

Antidifferentiate $v(t)$, add a constant, make p into a function of t.

```
> C2:=solve(p(0)=1120,C2);
```

$$C2 := 1120$$

```
> p(t);
```

$$-16\,t^2 + 48\,t + 1120$$

```
> t1:=solve(p(t)=0,t);
```

$$t1 := -7, 10$$

```
> t1:=t1[2];
```

$$t1 := 10$$

```
> v(t1);
```

$$-272$$

We see that the first solution to $p(t) = 0$ is negative, which is impossible, so we select the second solution, which is positive. The stone hits the ground 10 seconds after it is tossed upwards, striking the ground at 272 feet per second *downwards*.

C1M13 Problems: Use Maple to solve the problems and plot the graphs.

1. A stone is thrown *downwards* from a cliff with a velocity of 48 feet per second and it lands 1120 feet below. Using the point where it lands as the reference point, determine when the stone lands and how fast it is going.

2. Find an antiderivative F of $f(x) = \arctan(x)$ that satisfies $F(1) = \pi$.

3. Suppose that $g(x) = \sin(x) + e^{2x}$, G is an antiderivative of g, and H is an antiderivative of G. If $G(0) = 3$ and $H(0) = 5$, find $H(x)$ and $H(\pi)$.

C1M14

Integrals as Area Accumulators

Most textbooks do a good job of developing the integral and this is not the place to provide that development. We will show how Maple presents Riemann Sums and the accompanying diagrams and then focus on integrals from a to x. We hear about "moving the goalpost" when standards of performance are raised, but here that is exactly what is happening because x is a variable. We will be focusing on functions that are bounded on their domain, which will be an interval $[a,b]$, and are continuous at all except possibly a finite number of points. For this reason, when we break $[a,b]$ up into n subintervals we may assume that the subintervals all have the same length, $\dfrac{b-a}{n} = \Delta x$. In more general situations, we allow the subintervals to be of random length and then force the largest interval to get small as a means of controlling the approximation in the limit which we will call a definite integral. Let's break all this down into small parts and then assemble them in a useful way.

1. Suppose that f is a bounded function on $[a,b]$, f is continuous at all but at most a finite number of points, and n is a positive integer.

2. Let $\Delta x = \dfrac{b-a}{n}$. Then there are $n+1$ points $\{x_i\}_0^n$ determined by
$$x_0 = a = a + 0 \cdot \Delta x,\ \ x_1 = a + 1 \cdot \Delta x,\ \ x_2 = a + 2 \cdot \Delta x, \ldots,\ \ x_i = a + i \cdot \Delta x, \ldots,\ \ x_n = a + n \cdot \Delta x = b$$

3. In each subinterval $[x_{i-1}, x_i]$, a value t_i is selected. Obviously $x_{i-1} \le t_i \le x_i$.

4. Intuitively, we think of $f(t_i) \cdot \Delta x$ as the area of a rectangle of height $f(t_i)$ and base with width Δx placed above the interval $[x_{i-1}, x_i]$.

5. The value $\displaystyle\sum_{n=1}^{\infty} f(t_i) \cdot \Delta x$ is called a *Riemann Sum* and its value provides an approximation to the signed area between $y = f(x)$ and the x-axis.

6. We write $\displaystyle\lim_{n\to\infty} \sum_{n=1}^{\infty} f(t_i) \cdot \Delta x \ = \int_a^b f(x)\,dx$

7. On subintervals where $f(x) < 0$, there will be a negative contribution, so we must be careful how we use the phrase "the integral of f is the area beneath the curve". It makes sense for *positive* f only.

Maple Examples: Maple has three plotting commands `leftbox, middlebox, rightbox` that illustrate Riemann sums. It also has three numerical commands that compute Riemann sums, `leftsum, middlesum, rightsum`. We will illustrate their use for x^2 on the interval $[1,3]$ using 13 subintervals. These commands are all in the package `student`.

```
> restart:    with(plots):    with(student):
> leftsum(x^2,x=1..3,13);
```

$$\frac{2}{13}\left(\sum_{i=0}^{12}\left(1+\frac{2}{13}i\right)^2\right)$$

```
> value(leftsum(x^2,x=1..3,13));
```

$$\frac{1362}{169}$$

```
> evalf(leftsum(x^2,x=1..3,13));
```

$$8.059171595$$

```
> rightsum(x^2,x=1..3,13);
```

$$\frac{2}{13}\left(\sum_{i=1}^{13}\left(1+\frac{2}{13}i\right)^2\right)$$

```
> value(rightsum(x^2,x=1..3,13));
```

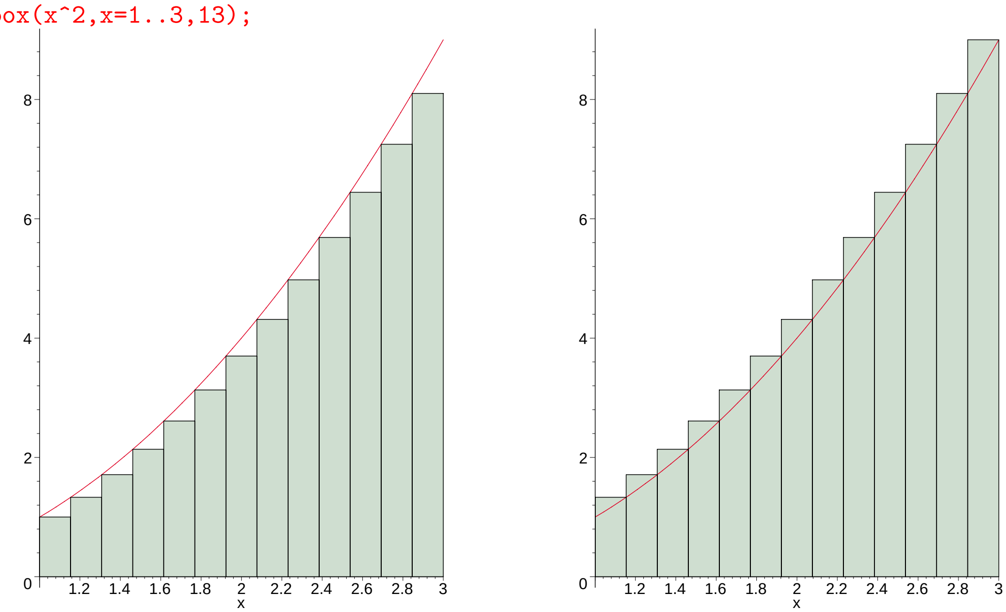
$$\frac{1570}{169}$$

```
> evalf(rightsum(x^2,x=1..3,13));
```

$$9.289940825$$

Leftbox is on the left and rightbox is on the right.

```
> leftbox(x^2,x=1..3,13);
```



Maple Animation Example: We will use the function $f(x) = x\,\sin(\pi x)$ on $[0, 3]$ and set up an animation of the approximation of $\int_0^3 x\,\sin(\pi x)\,dx$ using `middlebox`. The reader is **urged** to type in the commands below or to copy and paste, and to watch the animation. The value of each Riemann sum using the midpoint of the subinterval is shown near the top of the display.

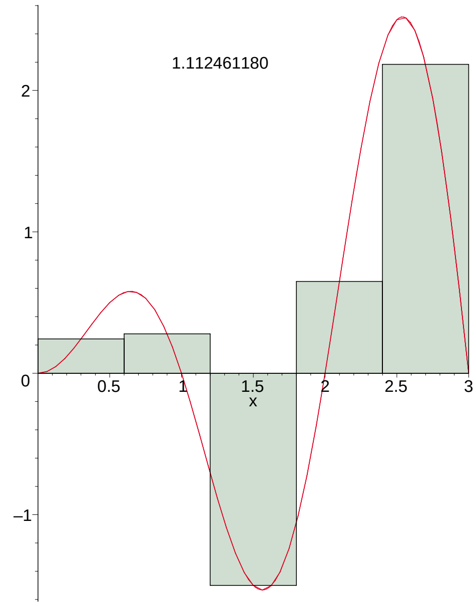
```
> restart:    with(plots):
> f:=x->x*sin(Pi*x);
```

$$f := x \to x\,\sin(\pi x)$$

```
> nstart:=5;    frameno:=50;
```

$$nstart := 5$$
$$frameno := 50$$

```
> framenumbers:=[seq(nstart+i,i=0..(frameno-1))]:
> A:=display(seq(middlebox(f(x),x=0..3,i),i=framenumbers),insequence=true):
> B:=animate(f(x),x=0..3,y=-2..3,color=red, frames=frameno):
> C:=display(seq(textplot([1.3,2.2,evalf(middlesum(f(x),x=0..3,i))]),i=5..(frameno+4)),
  insequence=true):
> display(A,B,C);
```



Let's look at the last approximation and the actual answer in floating point. By using uppercase "I" in `Int` on the left we obtain an inert integral, while the lowercase "i" yields an active command "integrate it" on the right in `int`.

```
> evalf(middlesum(f(x),x=0..3,frameno+4));
```

$$.9561427706$$

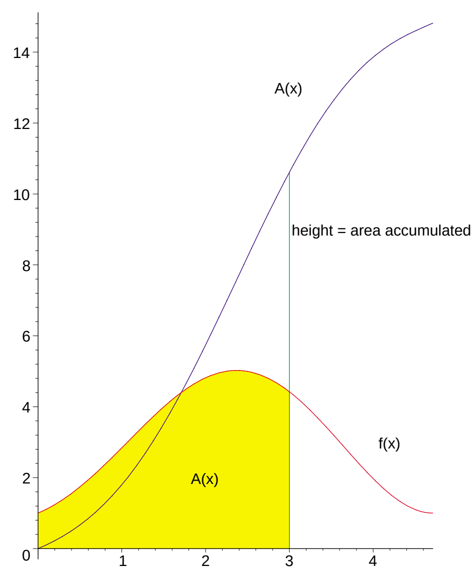
```
> Int(f(x),x=0..3)=evalf(int(f(x),x=0..3));
```

$$\int_0^3 x\,\sin(\pi x)\,dx = .9549296583$$

Play the animation! Click on the display. A box will appear around the figure and a "tape player" window will appear in the *context bar*. Click on the "Play" button and watch!

♠ It is important to note that Maple deals with definite integrals by simply including the range. For example, the value of $\int_a^b f(x)\,dx$ results from `int(f(x),x=a..b)`.

Now we will turn our attention to $f(x)$ and $A(x) = \displaystyle\int_a^x f(t)\,dt$. This integral intuitively 'accumulates area' as x moves from left to right.



We will state the first part of the **Fundamental Theorem of Calculus** and then illustrate it using a function defined in a piecewise manner.

The Fundamental Theorem of Calculus (Part One): Assume that f is continuous on $[a, b]$ and that

the function A is defined by $A(x)=\int_a^x f(t)\,dt$ for $a \le t \le b$. Then, $A'(x)=f(x)$ for all x in (a,b). In other words, A is *an* antiderivative for f.
Sometimes we write

$$D_x\left(\int_a^x f(t)\,dt\right)=f(x)$$

Maple Example: Suppose that f is defined by:

$$f(x)=\begin{cases} 1-x, & \text{if } x\le 1 \\ \ln(x), & \text{if } 1<x<e \\ (x-e-1)^2, & \text{if } e\le x \end{cases}$$

We will plot the graph of $f(x)$ and of $A(x)=\int_0^x f(t)\,dt$, which of course is the area accumulator function for $f(x)$. We will note that f is continuous, but is not differentiable at $x=1$ and $x=e$. It will be important to observe that the graph of $A(x)$ is smooth at all points, indicating that A is differentiable everywhere, as it should be.

```
> restart:    with(plots):  with(student):    with(plottools):
> e:=exp(1);
```

$$e := \mathbf{e}$$

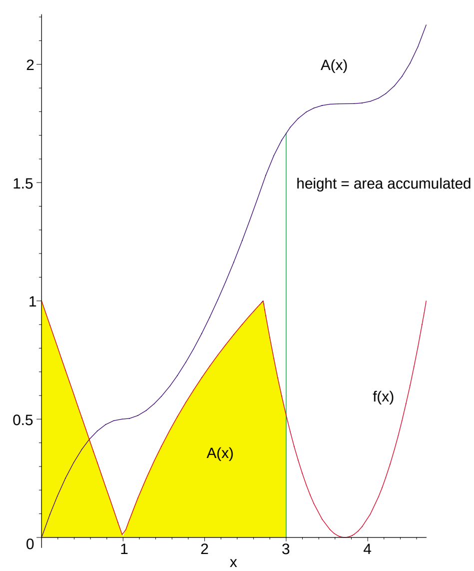
```
> f:=x->piecewise(x<=1,1-x,x>1 and x<e,ln(x),x>=e,(x-e-1)^2);
```

$$f := x \to \text{piecewise}(x\le 1, 1-x, 1<x \textbf{ and } x<e, \ln(x), e\le x, (x-e-1)^2)$$

```
> A:=x->int(f(t),t=0..x);
```

$$A := x \to \int_0^x f(t)\,dt$$

```
> A1:=plot(f(x),x=0..(e+2),color=red):
> A2:=plot(A(x),x=0..(e+2),color=blue):
> A3:=plot(f(x),x=0..3,color=yellow,filled=true):
> A4:=line([3,0],[3,A(3)],color=green):
> A5:=textplot([3.6,2,"A(x)"]):
> display(A1,A2,A3,A4,A5);
```



Without fanfare we will state the rest of the Fundamental Theorem of Integral Calculus. The same hypotheses still apply.

Fundamental Theorem of Calculus (Part Two): If F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

Suppose that f is continuous on $[a,b]$ and that $F(x) = \int_a^x f(t)\,dt$. If we apply the Mean Value Theorem to F on $[a,b]$, then there is a point c in (a,b) for which

$$\frac{F(b)-F(a)}{b-a} = F'(c)$$
$$\frac{\int_a^b f(t)\,dt - \int_a^a f(t)\,dt}{b-a} = f(c)$$
$$\frac{1}{b-a}\int_a^b f(t)\,dt = f(c)$$

This last value is sometimes called the *average value of* $f(x)$ *over* $[a,b]$. We know that $F' = f$ and $\int_a^a g(x)\,dx = 0$ for any function g and these facts were used above. We summarize this as the

Mean Value Theorem for Integrals: If f is continuous on $[a,b]$, then there is a number c in $[a,b]$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$.

Maple Example: Find a value c that satisfies the MVT for integrals for $f(x) = 3\,e^{-x} + x\,\sin(\pi x)$ on the interval $[0,3]$ and display a graph that illustrates this theorem.

```
> restart:    with(plots):    with(plottools):
> f:=x->3*exp(-x)+x*sin(Pi*x);
```

$$f := x \to 3\,\mathbf{e}^{-x} + x\,\sin(\pi x)$$

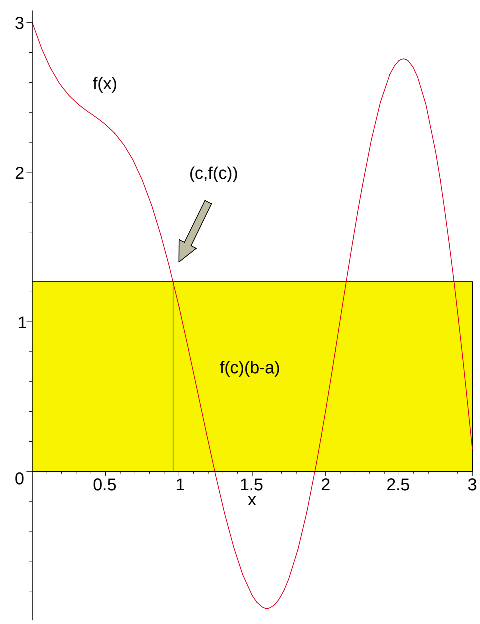
```
> A:=int(f(x),x=0..3);
```

$$A := -3\frac{(\mathbf{e}^{(-3)}\pi - 1 - \pi)}{\pi}$$

```
> c:=fsolve(f(x)=A/3,x,0..3);
```

$$c := .9601184662$$

```
> A1:=plot(f(x),x=0..3,color=red):
> A2:=plot(f(c),x=0..3,color=blue):
> A3:=plot(f(c),x=0..3,color=yellow,filled=true):
> A4:=line([c,0],[c,f(c)],color=green):
> A5:=textplot([.5,2.6,"f(x)"]):
> A6:=line([3,0],[3,f(c)],color=black):
> A7:=arrow([1.2,1.8],[1,1.4],.05,.13,.3,color=khaki):
> A8:=textplot([1.25,2,"(c,f(c))"]):
> A9:=textplot([1.5,.7,"f(c)(b-a)"]):
> display(A1,A2,A3,A4,A5,A6,A7,A8,A9);
```



C1M14 Problems: Use Maple to solve the problems and plot the graphs.

1. For $f(x) = 3\,e^{-x}\,sin(x)$ on $[0, \pi]$, evaluate (use `evalf`) and display graphically the the left, right, and middle sums with 47 subintervals. Remember that the commands you will need are in `student`.

2. Define $g(x) = \dfrac{1}{x}$ for x in $[1/4,\, 5]$. Then define $G(x) = \int_1^x g(t)\,dt$. This is how $\ln(x)$ is defined in some textbooks when exponentials and logarithms are delayed until after the integral has been developed. Display g and G on the same graph and fill the graph below $g(x)$ from 1 to 3.

3. For $f(x) = x\,\sin(x^2)$ on $[0, \pi]$, use Maple to find the average value of f on this interval and display a graph that illustrates the Mean Value Theorem for Integrals.